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On the horizontal gauge cohomology and non-removability of the spectral parameter

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Abstract. Given a zero-curvature representation α, the horizontal gauge cohomology group $\bar{H}^1_\alpha$ is shown to contain obstructions to removability of the "spectral parameter" of α. A method to compute $\bar{H}^1_\alpha$ is suggested.

Introduction

Important integration techniques ([1, 6, 8, 9]) for soliton equations, especially in dimension two, involve a zero-curvature representation (ZCR) taking values in a non-solvable Lie algebra $\mathfrak{g}$ and depending on a non-removable "spectral" parameter ([2, 26]). This is why one-parametric families of ZCR's, also referred to as "linear problems", are often considered as attributes of "complete integrability" of nonlinear partial differential equations (PDE). From another perspective, ZCR's are linear, finite-dimensional coverings [15], and often realizations of the Wahlquist and Estabrook [25] prolongation structures. Yet another interpretation has been given in terms of pseudospherical surfaces [19, 10]. Relations to the geometry of surfaces are also the subject of [5, 20, 21].

In [16], using the so-called first gauge cohomology, a characteristic element χ_α was associated to any ZCR α of a formally integrable system of nonlinear PDE's; the case of $\mathfrak{g} = \mathfrak{sl}_2$ being further developed in [17]. Independently, in his treatment of "continual classes" of evolution equations possessing a ZCR, Sakovich [18] introduced essentially the same concept, although without cohomological interpretation.

In this paper we deal with zeroth (horizontal) gauge cohomology $\bar{H}^q_\alpha$ associated with any ZCR α (Section 1). For ZCR's depending on a parameter we introduce a horizontal cohomology class $[\dot{\alpha}] \in \bar{H}^1_\alpha$ that is an obstruction to removability of the parameter (Section 2). This paper is to a great extent parallel to the work by Krasil'shchik and Igonin [13], who study general coverings depending on a parameter, with the so-called $\mathcal{C}$-cohomology in the background (Krasil'shchik [11, 12]).

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ZCR's related to pseudospherical surfaces need not depend on a parameter, but when they do, infinite-dimensional sequences of conservation laws are generated ([19, 5]), provided the parameter is not removable. We discuss in detail an example [21, Ch. I, Sect. 7.10], which features a removable parameter.

In Section 4 we show a simple way to compute $\bar H^1_\alpha$, applicable to typical equations of mathematical physics. The idea is to convert the problem into one already solved—finding a ZCR, although with coefficients in a larger algebra.

It should be noted that when the Lie algebra $\mathfrak g$ is one-dimensional, then the horizontal gauge cohomology becomes the horizontal cohomology $\bar H^q = E_1^{0,q}$ of Vinogradov [23] and is effectively computable by means of the $\mathcal C$-spectral sequence [23, 22]. Also, according to Verbovetsky [24], for general $\mathfrak g$ we still have $\bar H^q_\alpha = E_1^{0,q}$ for a generalized $\mathcal C$-spectral sequence with coefficients in a $\mathcal C$-module, which leads to a computation method for the terms $E_1^{p,q}$ with $p \ge 1$, but not the terms $E_1^{0,q} = \bar H^q_\alpha$ we need here.

1. Preliminaries

We recall briefly the basics of geometry of PDE's from [14, 23], which are the main references for all concepts unexplained below. All manifolds, functions, vector fields etc. are assumed to be smooth of class C^∞. Let X be an m-dimensional manifold, $1 < m < \infty$, let $Y \to X$ be a fibered manifold over X of fibre dimension $h \le \infty$. Consider the infinite jet prolongation $j^\infty Y \to X$ of $Y \to X$. Local coordinates x^i, $i = 1, \dots, m$, on X and u^k, $k = 1, \dots, h$, along the fibres of $Y \to X$ induce natural local coordinates u^k_I along the fibres of $j^\infty Y \to Y$; here and in what follows I stands for a symmetric multi-index in the i's. The commuting vector fields $D_i = \partial/\partial x^i + \sum_{r,I} u^k_{Ii}\,\partial/\partial u^k_I$ on $j^\infty Y$ are called *total derivatives*. Differential operators D_I acting on functions on $j^\infty Y$ are defined recursively by $D_{Ii} = D_I \circ D_i$.

In this framework we consider a finite system of finite-order PDE's

$$F^l(x^i, u, \dots, u^k_I, \dots) = 0, \tag{1}$$

$l = 1, \dots, n$, on unknowns u^k and their derivatives u^k_I. We assume that system (1), along with all its differential consequences $D_I F^l = 0$, determines a submanifold $\mathcal E \subseteq j^\infty Y$ (typically infinite-dimensional). The total derivatives are tangent to $\mathcal E$, hence they have a well-defined action on smooth functions on $\mathcal E$. The restricted fields $D_i = D_i|_{\mathcal E}$ span the so-called *Cartan distribution* on $\mathcal E$, denoted $\mathcal C$. The structure $(\mathcal E, \mathcal C)$ is a particular instance of a *diffiety* (see loc. cit.).

On $\mathcal E$, we have the direct decomposition of the tangent bundle $T\mathcal E$ as $\mathcal C \oplus V\mathcal E$, the Cartan distribution and the vertical vector bundle with respect to the projection on X. Let $C^\infty\mathcal E$ denote the ring of C^∞ functions on $\mathcal E$, let $\Lambda^{1,0}\mathcal E = \operatorname{Ann}\mathcal C$ and $\Lambda^{0,1}\mathcal E = \operatorname{Ann} V\mathcal E$ denote the $C^\infty\mathcal E$-modules of contact 1-forms and horizontal 1-forms, respectively. Denoting by $\Lambda^r\mathcal{E}$ the $C^\infty\mathcal{E}$-modules of antisymmetric r-forms on $\mathcal{E}$, we have the induced splittings $\Lambda^r\mathcal{E} = \bigoplus_{p+q=r} \Lambda^{p,q}\mathcal{E}$ into $r+1$ direct summands $\Lambda^{p,q}\mathcal{E} = \bigwedge^p \Lambda^{1,0}\mathcal{E} \wedge \bigwedge^q \Lambda^{0,1}\mathcal{E}$. Accordingly, the exterior differentials split into the sum $d = \bar{d} + \ell$ of the *horizontal* differential $\bar{d}\colon \Lambda^{p,q}\mathcal{E} \to \Lambda^{p,q+1}\mathcal{E}$ and the *vertical* differential $\ell\colon \Lambda^{p,q}\mathcal{E} \to \Lambda^{p+1,q}\mathcal{E}$. With respect to $\bar{d}$ and ℓ, the collection of the $C^\infty\mathcal{E}$-modules $\Lambda^{p,q}\mathcal{E}$, $p, q \ge 0$, is the *variational bicomplex*, whose associated spectral sequence (with $E_0^{pq} = \Lambda^{p,q}$) is the *$\mathcal{C}$-spectral sequence*. Coordinate formulas for the differentials $\bar{d}, \ell$ are derived from their action on $\Lambda^{0,0}\mathcal{E} = C^\infty\mathcal{E}$, which is

$$\bar{d}f = \sum_i D_i f\, dx^i,$$

$$\ell f = \sum_{k,I} \frac{\partial f}{\partial u_I^k}\omega_I^k, \qquad \omega_I^k = du_I^k - \sum_i u_{iI}^k\, dx^i.$$

Turning to the material of [16], we consider a finite-dimensional real or complex Lie algebra $\mathfrak{g}$. Its tensor product with the graded exterior algebra $\bar{\Lambda}\mathcal{E} = \bigoplus_q \bar{\Lambda}^q\mathcal{E}$ is a graded nonassociative algebra under the bracket $[A\mu, B\nu] = [A,B]\mu \wedge \nu$ for $A, B \in \mathfrak{g}$ and $\mu, \nu \in \bar{\Lambda}\mathcal{E}$. Then

$$[\rho,\sigma] = -(-1)^{rs}[\sigma,\rho], \qquad \bar{d}[\rho,\sigma] = [\bar{d}\rho,\sigma] + (-1)^r[\rho,\bar{d}\sigma]$$

for $\rho \in \bar{\Lambda}^r\mathcal{E}\otimes\mathfrak{g}$, $\sigma \in \bar{\Lambda}^s\mathcal{E}\otimes\mathfrak{g}$. It is technically convenient to assume that $\mathfrak{g}$ is a matrix algebra (the assumption being irrestrictive by the Ado theorem [3]), i.e., that $\mathfrak{g}$ is a subalgebra in some $\mathfrak{gl}_n$. Then $\bar{\Lambda}\mathcal{E}\otimes\mathfrak{gl}_n$ is a graded associative algebra with respect to the multiplication $A\mu \cdot B\nu = (A\cdot B)\mu\wedge\nu$ induced by the ordinary matrix multiplication, while

$$[\rho,\sigma] = \rho\cdot\sigma - (-1)^{rs}\sigma\cdot\rho, \qquad \bar{d}(\rho\cdot\sigma) = \bar{d}\rho\cdot\sigma + (-1)^r\rho\cdot\bar{d}\sigma$$

for $\rho \in \bar{\Lambda}^r\mathcal{E}\otimes\mathfrak{gl}_n$, $\sigma \in \bar{\Lambda}^s\mathcal{E}\otimes\mathfrak{gl}_n$. Elements of $C^\infty\mathcal{E}\otimes\mathfrak{g}$ will be called *$\mathfrak{g}$-matrices*.

A $\mathfrak{g}$-valued *zero-curvature representation* (ZCR) for $\mathcal{E}$ is a horizontal 1-form $\alpha \in \bar{\Lambda}^1\mathcal{E}\otimes\mathfrak{g}$ satisfying

$$\bar{d}\alpha = \tfrac{1}{2}[\alpha,\alpha]. \tag{2}$$

If $\alpha = \sum_i A_i\, dx^i$, $A_i \in \mathfrak{g}$, then eq. (2) becomes $D_j A_i - D_i A_j + [A_i, A_j] = 0$, for pairs $\{i,j\}$ such that $i \ne j$.

Given a ZCR α, we consider operators

$$\bar{\partial}_\alpha = \bar{d} - \operatorname{ad}_\alpha\colon \bar{\Lambda}^q\mathcal{E}\otimes\mathfrak{g} \to \bar{\Lambda}^{q+1}\mathcal{E}\otimes\mathfrak{g},$$

where $\operatorname{ad}_\alpha\rho = [\alpha,\rho]$ for any $\rho \in \bar{\Lambda}\mathcal{E}\otimes\mathfrak{g}$. We have $\bar{\partial}_\alpha\circ\bar{\partial}_\alpha = 0$ as a consequence of (2), which gives the *horizontal gauge complex* (or 0-th linear gauge complex [16])

$$0 \to \bar{\Lambda}^0\mathcal{E}\otimes\mathfrak{g} \xrightarrow{\bar{\partial}_\alpha} \bar{\Lambda}^1\mathcal{E}\otimes\mathfrak{g} \xrightarrow{\bar{\partial}_\alpha} \bar{\Lambda}^2\mathcal{E}\otimes\mathfrak{g} \to \cdots \to \bar{\Lambda}^m\mathcal{E}\otimes\mathfrak{g} \to 0.$$

The groups

$$\bar{H}^q_\alpha(\mathcal{E},\mathfrak{g}) = \frac{\operatorname{Ker}\bigl(\bar\Lambda^q\mathcal{E}\otimes\mathfrak{g} \xrightarrow{\bar\partial_\alpha} \bar\Lambda^{q+1}\mathcal{E}\otimes\mathfrak{g}\bigr)}{\operatorname{Im}\bigl(\bar\Lambda^{q-1}\mathcal{E}\otimes\mathfrak{g} \xrightarrow{\bar\partial_\alpha} \bar\Lambda^{q}\mathcal{E}\otimes\mathfrak{g}\bigr)}$$

are called the *horizontal gauge cohomology groups* with respect to the ZCR α.

If $\alpha = \sum_i A_i\,dx^i$, $A_i \in \mathfrak{g}$, then for an arbitrary $\mathfrak{g}$-matrix $C \in C^\infty\mathcal{E}\otimes\mathfrak{g}$ we have

$$\bar\partial_\alpha C = \sum_i \widehat{D}_i C\,dx^i, \qquad \widehat{D}_i C = D_i C - [A_i, C]. \tag{3}$$

Operators $\widehat{D}_i$ commute whenever α is a ZCR. We define recursively $\widehat{D}_{Ii} = \widehat{D}_I \circ \widehat{D}_i$.

Likewise, for $p = 1$ the corresponding modules $\Lambda^{1,q}\otimes\mathfrak{g}$ form the so-called *first linear gauge complex*

$$0 \to \Lambda^{1,0}\mathcal{E}\otimes\mathfrak{g} \xrightarrow{\bar\partial_\alpha} \Lambda^{1,1}\mathcal{E}\otimes\mathfrak{g} \to \cdots \to \Lambda^{1,m}\mathcal{E}\otimes\mathfrak{g} \to 0.$$

The groups

$$H^{1,q}_\alpha(\mathcal{E},\mathfrak{g}) = \frac{\operatorname{Ker}\bigl(\Lambda^{1,q}\mathcal{E}\otimes\mathfrak{g} \xrightarrow{\bar\partial_\alpha} \bar\Lambda^{1,q+1}\mathcal{E}\otimes\mathfrak{g}\bigr)}{\operatorname{Im}\bigl(\bar\Lambda^{1,q-1}\mathcal{E}\otimes\mathfrak{g} \xrightarrow{\bar\partial_\alpha} \bar\Lambda^{1,q}\mathcal{E}\otimes\mathfrak{g}\bigr)}$$

are computable in principle. We refer to [16] or [24] for details of the isomorphism $H^{1,q}_\alpha(\mathcal{E},\mathfrak{g}) \cong \operatorname{Ker}(\mathfrak{g}\otimes\widehat{P}_{m-q} \to \mathfrak{g}\otimes\widehat{P}_{m-q-1})/\operatorname{Im}(\mathfrak{g}\otimes\widehat{P}_{m-q+1} \to \mathfrak{g}\otimes\widehat{P}_{m-q})$. Here the starting point is the "compatibility complex" $P_0 \to P_1 \to P_2 \to \cdots$, where P_i are modules of sections of appropriate vector bundles over $\mathcal{E}$ (see [24]). We only remark here that $\dim P_0 = h =$ the number of unknowns, $\dim P_1 = n =$ the number of equations in system (1), and the first arrow $P_0 \to P_1$ can be identified with the operator of *universal linearization*, while each of the subsequent arrows expresses the integrability conditions for the previous one. Always $P_j = 0$ for all $j > m$, while for non-overdetermined systems we have even $P_j = 0$ for all $j > 1$. The dual complex $\cdots \to \widehat{P}_2 \to \widehat{P}_1 \to \widehat{P}_0$ is composed of formally adjoint operators between dual modules $\widehat{P} := \operatorname{Hom}(P, \bar\Lambda^m\mathcal{E})$. Finally,

$$\cdots \to \mathfrak{g}\otimes\widehat{P}_2 \to \mathfrak{g}\otimes\widehat{P}_1 \to \mathfrak{g}\otimes\widehat{P}_0$$

is obtained by replacing each operator D_i with its covariant counterpart $\widehat{D}_i = D_i - \operatorname{ad}_{A_i}$.

The element $\ell(\alpha) \in \Lambda^{1,1}\mathcal{E}\otimes\mathfrak{g}$ is a cocycle by (2); the corresponding 1-st cohomology class $\operatorname{ch}(\alpha) = [\ell(\alpha)] \in H^{1,1}_\alpha(\mathcal{E},\mathfrak{g})$ will be called the *characteristic class* of the ZCR α. The *characteristic element* χ_α we introduced in [16] is the image of $\operatorname{ch}(\alpha)$ under the above isomorphism

$$H^{1,1}_\alpha(\mathcal{E},\mathfrak{g}) \cong \frac{\operatorname{Ker}(\mathfrak{g}\otimes\widehat{P}_{m-1} \to \mathfrak{g}\otimes\widehat{P}_{m-2})}{\operatorname{Im}(\mathfrak{g}\otimes\widehat{P}_{m} \to \mathfrak{g}\otimes\widehat{P}_{m-1})}.$$

In [16, Prop. 4.2] we proved that if $H^{2,0}_\alpha = 0$ and $\operatorname{ch}(\alpha) = 0$, then $\alpha = \bar d\theta \cdot \theta^{-1}$ for an appropriate G-matrix θ (such α's are called *trivial*).

Assuming system (1) non-overdetermined, P_2 is zero. Then $H^{1,1}_\alpha(\mathcal{E}, \mathfrak{g})$ vanishes for $m > 2$ and every ZCR is trivial, while for $m = 2$ we have an isomorphism $H^{1,1}_\alpha(\mathcal{E}, \mathfrak{g}) \cong \operatorname{Ker}(\mathfrak{g} \otimes \widehat{P}_1 \to \mathfrak{g} \otimes \widehat{P}_0)$ and characteristic elements are n-tuples of $\mathfrak{g}$-matrices χ_l defined on $\mathcal{E}$ and satisfying

$$(4) \qquad \sum_{l,J} (-1)^{|J|} \widehat{D}_J \left(\frac{\partial F^l}{\partial u^k_J} \chi_l \right) = 0, \qquad \widehat{D}_i = D_i - \operatorname{ad}_{A_i}.$$

We also have an explicit formula for the characteristic element for α, first written by Sakovich [18] for evolution equations: if $\mathfrak{g}$-matrices C^J_l satisfy

$$\bar d\alpha - \tfrac{1}{2}[\alpha, \alpha] = \sum_{l,J} D_J F^l \cdot C^J_l,$$

then

$$(5) \qquad \chi_l = \sum_J (-\widehat{D})_J C^J_l \big|_{\mathcal{E}}.$$

For a matrix function $S \colon \mathcal{E} \to G$, we have the conjugation $\operatorname{Ad}_S \colon \bar\Lambda^q \mathcal{E} \otimes \mathfrak{g} \to \bar\Lambda^q \mathcal{E} \otimes \mathfrak{g}$ defined by $\gamma \mapsto S \cdot \gamma \cdot S^{-1}$. For any ZCR α, the form

$$\alpha^S = \bar d S \cdot S^{-1} + S \cdot \alpha \cdot S^{-1}$$

is another ZCR; we call it *gauge equivalent* to α. One easily checks that

$$(6) \qquad \bar\partial_{\alpha^S} \circ \operatorname{Ad}_S = \operatorname{Ad}_S \circ \bar\partial_\alpha,$$

so that Ad_S is a morphism of the horizontal gauge complexes. Since Ad_S is invertible (with inverse $\operatorname{Ad}_{S^{-1}}$), we have

$$(7) \qquad \bar H^q_\alpha(\mathcal{E}, \mathfrak{g}) \cong \bar H^q_{\alpha^S}(\mathcal{E}, \mathfrak{g}).$$

Similarly to (7), we have the isomorphism $H^{1,q}_\alpha(\mathcal{E}, \mathfrak{g}) \cong H^{1,q}_{\alpha^S}(\mathcal{E}, \mathfrak{g})$ induced by the conjugation Ad_S. Hence, gauge equivalent ZCR's have conjugate characteristic elements. The converse is not true in general and two ZCR's with conjugate characteristic elements may still be gauge inequivalent. Examples are provided by numerous completely integrable equations of mathematical physics (e.g., KdV, mKdV, sine-Gordon, etc.) that have their characteristic elements independent of the spectral parameter.

We close this section a result easy to prove:

Proposition 1. *Let $\alpha \in \bar\Lambda^1(\mathcal{E}, \mathfrak{g})$ be a $\mathfrak{g}$-valued ZCR on $\mathcal{E}$, with $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$, let $\alpha = \alpha' + \alpha''$ with $\alpha' \in \bar\Lambda^1(\mathcal{E}, \mathfrak{g}')$, $\alpha'' \in \bar\Lambda^1(\mathcal{E}, \mathfrak{g}'')$. Then $\bar H^q_\alpha = \bar H^q_{\alpha'} \oplus \bar H^q_{\alpha''}$.*

2. Removability

In this section we consider a ZCR depending on a parameter λ and show that the first horizontal gauge cohomology group $\bar{H}^1$ contains obstructions to removability of λ. Let α_{λ_0} be a member of a smooth family of ZCR's

$$\alpha_\lambda \in \bar{\Lambda}^1\mathcal{E} \otimes \mathfrak{g}, \tag{8}$$

with λ taking values in an open interval containing λ_0. Denoting

$$\dot{\alpha}_{\lambda_0} = \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda_0} \alpha_\lambda \in \mathfrak{g},$$

we have

$$\bar{d}\dot{\alpha}_{\lambda_0} = \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda_0} \bar{d}\alpha_\lambda = \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda_0} (\alpha_\lambda \cdot \alpha_\lambda) = \dot{\alpha}_{\lambda_0} \cdot \alpha_{\lambda_0} + \alpha_{\lambda_0} \cdot \dot{\alpha}_{\lambda_0},$$

whence $\bar{\partial}_{\alpha_{\lambda_0}} \dot{\alpha}_{\lambda_0} = 0$. Thus,

Proposition 2. *The form $\dot{\alpha}_\lambda \in \bar{\Lambda}^1\mathcal{E} \otimes \mathfrak{g}$ is a 1-cocycle with respect to $\partial_{\alpha_\lambda}$.*

Recall that the parameter λ is *removable* if the forms α_λ are gauge equivalent for different values of λ, otherwise it is called *non-removable.* A parameter λ created by gauge action with respect to a matrix S_λ will be removed by the gauge action with respect to S_λ^{-1}.

Proposition 3. *The following statements about the family* (8) *are equivalent*:

(i) *the parameter λ is removable*;
(ii) *each of the forms $\dot{\alpha}_\lambda \in \bar{\Lambda}^1\mathcal{E} \otimes \mathfrak{g}$ is a coboundary, $\dot{\alpha}_\lambda = \bar{\partial}_{\alpha_\lambda} Q_\lambda$, with $\mathfrak{g}$-matrices Q_λ depending smoothly on λ.*

Proof. Let the matrix Lie algebra $\mathfrak{g}$ be associated to a connected and simply connected matrix Lie group G.

Assume (i) and fix λ_0. Then there exist G-matrices S_λ such that $\alpha_{\lambda_0}^{S_\lambda} = \alpha_\lambda$ and $S_{\lambda_0} = E =$ the identity matrix. Denote $\dot{S}_{\lambda_0} = \partial/\partial\lambda|_{\lambda=\lambda_0} S_\lambda$, which takes values in $\mathfrak{g}$, the tangent space to G at $S_{\lambda_0} = E$, the unit element. Then we have

$$\begin{aligned} 0 &= \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda_0} \alpha_{\lambda_0} = \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda_0} \alpha_\lambda^{S_\lambda^{-1}} = \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda_0} \left(-S_\lambda^{-1}\bar{d}S_\lambda + S_\lambda^{-1}\alpha_\lambda S_\lambda\right) \\ &= -\bar{d}\dot{S}_{\lambda_0} - \dot{S}_{\lambda_0}\alpha_{\lambda_0} + \alpha_{\lambda_0}\dot{S}_{\lambda_0} + \dot{\alpha}_{\lambda_0}, \end{aligned}$$

i.e., $\dot{\alpha}_{\lambda_0} = \bar{\partial}_{\alpha_{\lambda_0}} \dot{S}_{\lambda_0}$ for $\dot{S}_{\lambda_0} \in \bar{\Lambda}^0\mathcal{E} \otimes \mathfrak{g}$.

Conversely, let $\dot{\alpha}_\lambda = \bar{\partial}_{\alpha_\lambda} Q_\lambda$ for some $Q_\lambda \in C^\infty\mathcal{E} \otimes \mathfrak{g}$. Let S_λ be a G-matrix solution of the equation $\partial S_\lambda/\partial\lambda = Q_\lambda S_\lambda$ satisfying the initial condition $S_{\lambda_0} = E$. Considering the expression $Z_\lambda = \bar{d}S_\lambda + S_\lambda\alpha_{\lambda_0} - \alpha_\lambda S_\lambda = \big(\alpha_{\lambda_0}^{S_\lambda} - \alpha_\lambda\big)S_\lambda$, we have

$$\begin{aligned}\frac{\partial}{\partial\lambda}Z_\lambda &= \frac{\partial}{\partial\lambda}(\bar{d}S_\lambda + S_\lambda\alpha_{\lambda_0} - \alpha_\lambda S_\lambda)\\ &= (\bar{d}Q_\lambda - [\alpha_\lambda, Q_\lambda] - \dot\alpha_\lambda)S_\lambda + Q_\lambda(\bar{d}S_\lambda + S_\lambda\alpha_{\lambda_0} - \alpha_\lambda S_\lambda)\\ &= Q_\lambda Z_\lambda,\end{aligned}$$

while obviously $Z_{\lambda_0} = 0$. Therefore, $Z_\lambda = 0$ for all λ, which proves the assertion. □

Given a family α_λ of ZCR's, the following procedure yields the parameter-removing matrix S_λ^{-1} whenever it exists:

(1) Compute the elements $\dot\alpha_\lambda = \partial\alpha_\lambda/\partial\lambda$.
(2) If $\dot\alpha_\lambda$ are coboundaries, find $\mathfrak{g}$-matrices Q_λ such that $\dot\alpha_\lambda = \bar\partial_{\alpha_\lambda}Q_\lambda$.
(3) Solve the equation $\partial S_\lambda/\partial\lambda = Q_\lambda S_\lambda$ for a G-matrix S_λ under the initial condition $S_\lambda = E$.

For methods to perform the third step see, e.g., [7]. Concerning the second step, we shall show in the next section that H^1_α is effectively computable in principle. But before that we present an example.

Example 4. Sasaki [19] was probably the first to observe that equations possessing an $\mathfrak{sl}_2$-valued ZCR also describe pseudospherical surfaces, and suggested a procedure to derive infinite families of conservation laws thereof, assuming dependence of the ZCR on a parameter. A similar study has been independently initiated by Chern and Tenenblat [5] and continued in a series of papers, see the references in [21]. Needless to say, no infinite series of conservation laws will be generated in the case of removable parameter. Here and in the appendix below we complete the results of [4, 5] by showing that exactly this happens in the example of the Burgers equation.

Writing the Burgers equation as $u_t = u_{xx} + uu_x$, authors of [4] consider the ZCR

$$\begin{aligned}\alpha_\eta = A_\eta\,dx + B_\eta\,dt &= \begin{pmatrix}\frac12\eta & \frac14 u + \frac12\eta\\ \frac14 u - \frac12\eta & -\frac12\eta\end{pmatrix}dx\\ &\quad+\begin{pmatrix}\frac14\eta u & \frac14 u_x + \frac18 u^2 + \frac14\eta u\\ \frac14 u_x + \frac18 u^2 - \frac14\eta u & -\frac14\eta u\end{pmatrix}dt.\end{aligned}\tag{9}$$

For $\eta = 0$ the ZCR reduces to the conservation law $u\,dx + (u_x + \frac12 u^2)\,dt$. We shall perform the above three-step procedure to show that for all $\eta \ne 0$ the forms α_η are mutually gauge equivalent.

By differentiation we obtain

$$\dot\alpha_\eta = \begin{pmatrix}\frac12 & \frac12\\ -\frac12 & -\frac12\end{pmatrix}dx + \begin{pmatrix}\frac14 u & \frac14 u\\ -\frac14 u & -\frac14 u\end{pmatrix}dt.$$

For all $\eta \neq 0$ we solve $\dot\alpha = \bar\partial_\alpha Q = \bar d Q - \alpha Q + Q\alpha$ as

$$Q_\eta = \begin{pmatrix} 0 & -\frac12\eta^{-1} \\ -\frac12\eta^{-1} & 0 \end{pmatrix}$$

to show that each $\dot\alpha_\eta$ is a coboundary. For $\eta > 0$, the equation $\dot S_\eta = Q_\eta S_\eta$ has a solution

$$S_\eta = \begin{pmatrix} \frac12\eta^{-1/2} + \frac12\eta^{1/2} & \frac12\eta^{-1/2} - \frac12\eta^{1/2} \\ \frac12\eta^{-1/2} - \frac12\eta^{1/2} & \frac12\eta^{-1/2} + \frac12\eta^{1/2} \end{pmatrix}$$

in SL_2 satisfying the initial condition $S_1 = E$. Using S_η^{-1} as the gauge matrix, one transforms α_η to α_1, thus removing the parameter η. For $\eta < 0$ we similarly obtain a gauge equivalence between α_η and α_{-1}. But α_{-1} is transformable to α_1 via the gauge matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

This example is continued in the Appendix.

3. Computing the horizontal gauge cohomology

In the sequel we assume that the coefficient field is $\mathfrak{k} = \mathbb{R}$ or $\mathbb{C}$. Special attention is paid to Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$. Since $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus \mathfrak{k}$, we have

$$\bar H^q_\alpha(\mathcal{E}, \mathfrak{gl}_n) = \bar H^q_\alpha(\mathcal{E}, \mathfrak{sl}_n) \oplus \bar H^q \mathcal{E}$$

by Proposition 1. Here $\bar H^q$ stands for the usual ($\mathfrak{k}$-valued) horizontal cohomology.

In the case of $q = 0$ we have simply $\bar H^0_\alpha(\mathcal{E}, \mathfrak{g}) = \operatorname{Ker} \bar\partial_\alpha$. However, $\bar\partial_\alpha Q = \bar d Q - [\alpha, Q] = \sum_i \big(D_i Q - [A_i, Q]\big)\, dx^i = \widehat D_i Q\, dx^i$ for any $\mathfrak{g}$-matrix Q, and therefore the group $\bar H^0_\alpha(\mathcal{E}, \mathfrak{g})$ may be easily computed from the overdetermined system of linear equations in total derivatives

$$\widehat D_i Q = 0. \tag{10}$$

E.g., for trivial α we have $\bar H^0_\alpha(\mathcal{E}, \mathfrak{g}) \cong \bar H^0_0(\mathcal{E}, \mathfrak{g}) = \mathfrak{g} \otimes \operatorname{Ker} \bar d$, where $\operatorname{Ker} \bar d \cong \mathfrak{k}$ unless the diffiety $\mathcal{E}$ has a trivial factor (see [15]).

Since each $\widehat D_i$ is a $\mathfrak{k}$-linear differentiation, $\bar H^0_\alpha(\mathcal{E}, \mathfrak{gl}_n)$ is a $\mathfrak{k}$-algebra. We have the following interpretation for its invertible elements:

Proposition 5. *The set of all invertible elements in the $\mathfrak{k}$-algebra $\bar H^0_\alpha(\mathcal{E}, \mathfrak{gl}_n)$ coincides with the stabilizer of α with respect to the gauge action.*

Proof. Let $Q \in C^\infty\mathcal{E} \otimes \mathfrak{gl}_n$ be invertible. Then $\bar d Q = [\alpha, Q] = \alpha\cdot Q - Q\cdot\alpha$ if and only if $\alpha^Q = \bar d Q \cdot Q^{-1} + Q \cdot \alpha \cdot Q^{-1} = \alpha$. □

Remark 6. Nontrivial examples of ZCR's taken from the literature typically satisfy

$$\bar H^0_\alpha(\mathcal{E}, \mathfrak{gl}_n) \cong \mathfrak{k}, \qquad \bar H^0_\alpha(\mathcal{E}, \mathfrak{sl}_n) = 0. \tag{11}$$

Assuming that (11) holds, we propose the following way to determine the cohomology group $\bar{H}^1_\alpha(\mathcal{E},\mathfrak{sl}_n)$. In what follows G denotes the connected and simply connected Lie group associated to $\mathfrak{g}$. Denote by $\mathfrak{g}^\Delta$ the product $\mathfrak{g}\times\mathfrak{g}$ endowed with the bracket

$$[(A,A'),\ (B,B')] = ([A,B],\ [A',B]+[A,B']).$$

Then $\mathfrak{g}^\Delta$ is a Lie algebra—in fact, a semidirect product of the subalgebra $\{A'=0\}$ and the ideal $\{A=0\}$. Denote by G^Δ the product $G\times\mathfrak{g}$ endowed with the group multiplication

$$(A,A')\cdot(B,B') = (A\cdot B,\ A'+\operatorname{Ad}_A B'),$$

unit element being $(1,0)$, and the inverse

$$(A,A')^{-1} = (A^{-1}, -\operatorname{Ad}_{A^{-1}} A').$$

Then $\mathfrak{g}^\Delta$ is the Lie algebra of G^Δ. Given matrix realizations $\mathfrak{g}\subseteq\mathfrak{gl}_n$ and $G\subseteq\mathrm{GL}_n$, we henceforth use the matrix realizations

$$\mathfrak{g}^\Delta = \left\{ \begin{pmatrix} B & 0 \\ B' & B \end{pmatrix} \,\middle|\, B,B'\in\mathfrak{g} \right\} \subset \mathfrak{gl}_{2n}$$

and

$$G^\Delta = \left\{ \begin{pmatrix} A & 0 \\ A'A & A \end{pmatrix} \,\middle|\, A\in G,\ A'\in\mathfrak{g} \right\} \subset \mathrm{GL}_{2m}$$

for $\mathfrak{g}^\Delta$ and G^Δ.

The gauge action of a G^Δ-matrix $\tilde{A}=(A,A')$ on a $\mathfrak{g}^\Delta$-valued horizontal q-form $\tilde{\beta}=(\beta,\beta')\in\bar{\Lambda}^q\otimes\mathfrak{g}^\Delta$ is given by

$$(\beta,\beta')^{(A,A')} = (\beta^A,\ \bar{d}A'+[A',\beta^A]+\operatorname{Ad}_A\beta'). \tag{12}$$

Finally, we introduce the mapping

$$L_\alpha\colon \bar{\Lambda}^1\mathcal{E}\otimes\mathfrak{g}\to\bar{\Lambda}^1\mathcal{E}\otimes\mathfrak{g}^\Delta, \qquad L_\alpha\beta=(\alpha,\beta).$$

Proposition 7. (i) *A form $\beta\in\bar{\Lambda}^1\mathcal{E}\otimes\mathfrak{g}$ is a $\bar{\partial}_\alpha$-cocycle if and only if $L_\alpha\beta$ is a $\mathfrak{g}^\Delta$-valued ZCR.*

(ii) *Assuming* (11), *cocycles β_1,β_2 are cohomological if and only if $L_\alpha\beta_1, L_\alpha\beta_2$ are gauge equivalent with respect to the group G^Δ.*

Proof. Statement (i) immediately follows from the equality

$$\bar{d}L_\alpha\beta - L_\alpha\beta\cdot L_\alpha\beta = (\bar{d}\alpha-\alpha\cdot\alpha,\ \bar{\partial}_\alpha\beta).$$

(ii) Let $\beta_2=\beta_1+\bar{d}Q-[\alpha,Q]$ for some $Q\in C^\infty\mathcal{E}\otimes\mathfrak{g}$. Upon choosing $S=(E,Q)\in G^\Delta$, formula (12) yields $(L_\alpha\beta_1)^S=L_\alpha\beta_2$.

Conversely, let $S=(P,Q)\in G^\Delta$ exists such that $(L_\alpha\beta_1)^S=L_\alpha\beta_2$, i.e.,

$$(\alpha,\beta_2) = (\alpha^P, \bar{\partial}_{\alpha^P}Q+\operatorname{Ad}_P\beta_1) \tag{13}$$

by eq. (12). We see that P stabilizes α, hence $P\in\bar{H}^0_\alpha(\mathcal{E},\mathfrak{g})$ by Proposition 5. According to assumption (11) then $P=pE$ for some constant

$p \in \mathfrak{k}$, so that eq. (13) simplifies to $(\alpha, \beta_2) = (\alpha, \bar\partial_\alpha Q + \beta_1)$ and β_1, β_2 are cohomological. □

4. Normal forms

To convert Proposition 7 into an effective computational procedure, one further needs to develop algebraic classification of normal forms of characteristic elements in $\mathfrak{g}^\Delta$ in the spirit of [17, Prop. 3]. For $\mathfrak{g} = \mathfrak{sl}_2$ this is done below.

We assume that $m = 2$ and $\alpha = A\,dx + B\,dy$. The procedure to compute ZCR's of non-overdetermined systems (1) is as follows. Given eq. (1) and a Lie algebra $\mathfrak{g}$, solve the *determining system*

$$0 = D_y\bar A - D_x\bar B + [\bar A, \bar B], \tag{14}$$

$$0 = \sum_{J,l}(-1)^{|J|}\widehat D_J\left(\frac{\partial F^l}{\partial u^k_J}\bar\chi_l\right) \tag{15}$$

for unknowns $\bar\chi, \bar A, \bar B$ for $\bar\chi, \bar A, \bar B$ running through possible normal forms for triples (χ, A, B), $\chi \ne 0$, with respect to the group action

$$(\chi, A, B) \mapsto (S\chi S^{-1}, D_xSS^{-1} + SAS^{-1}, D_ySS^{-1} + SBS^{-1}).$$

The unknowns $\bar A, \bar B$ enter eqs. (14)–(15) via the coefficients of $\widehat D_x, \widehat D_y$; components of $\bar\chi$ are auxiliary unknowns. In general position we have $(n+1)\cdot\dim\mathfrak{g}$ equations for the same number of unknowns. Possible normal forms $\bar\chi, \bar A, \bar B$ depend only on the algebra $\mathfrak{g}$.

We consider the simplest case of $\mathfrak{sl}_2$ here, with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11}\end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & -b_{11}\end{pmatrix}.$$

A ZCR may degenerate to a nonlocal conservation law. E.g., if $a_{12}b_{21} = a_{21}b_{12}$, then $a_{11}\,dx + b_{11}\,dy$ is a conservation law; denoting h its potential, one easily checks that $\mathrm{e}^{-2h}(a_{12}\,dx + b_{12}\,dy)$ and $\mathrm{e}^{2h}(a_{21}\,dx + b_{21}\,dy)$ are two linearly dependent nonlocal conservation laws. Likewise, α degenerates (is gauge equivalent to the one that satisfies $a_{12}b_{21} = a_{21}b_{12}$) whenever at least one of the off-diagonal coefficients $a_{12}, a_{21}, b_{12}, b_{21}$ is zero (see [17]).

Let $C = \chi_1$ or any other χ_i that is nonzero (if all are zero, then α is trivial). Assuming α non-degenerate, by the main result of [17] we have exactly two possible normal forms for the matrices C, A:

– Nilpotent case

$$C = \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}, \quad A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0\end{pmatrix}. \tag{16}$$

– Diagonal case

$$C = \begin{pmatrix} r & 0 \\ 0 & -r\end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ 1 & -a_{11}\end{pmatrix}. \tag{17}$$

In general, no further reduction (of B or any other χ_i) is possible.

With Proposition 7 in mind, let us consider $\mathfrak{sl}_2^\Delta$-matrices of the block form

$$\tilde{C} = \begin{pmatrix} C & 0 \\ C' & C \end{pmatrix}, \qquad \tilde{A} = \begin{pmatrix} A & 0 \\ A' & A \end{pmatrix},$$

where C, A were introduced above and C', A' are arbitrary $\mathfrak{sl}_2$-matrices.

Proposition 8. *Assuming α nontrivial and non-degenerate, possible normal forms for the matrices $\tilde{C}$, $\tilde{A}$ are*

– *Nilpotent case*

$$\tilde{C} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ 0 & a_{32} & 0 & a_{12} \\ a_{41} & 0 & a_{21} & 0 \end{pmatrix}. \tag{18}$$

– *Diagonal case*

$$\tilde{C} = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & -r & 0 & 0 \\ s & 0 & r & 0 \\ 0 & -s & 0 & -r \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 1 & -a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & a_{12} \\ 0 & -a_{31} & 1 & -a_{11} \end{pmatrix}. \tag{19}$$

Proof. Provided $r \neq 0$ and $a_{12} \neq 0$, both covered by our assumptions, it is easy to find a matrix $\tilde{S} = (E, S') \in \mathrm{SL}_2^\Delta$ that transforms $\tilde{C}$ to its normal form by conjugation and then a matrix $\tilde{T} = (E, T') \in \mathrm{SL}_2^\Delta$ that transforms $\tilde{A}$ to its normal form by gauge equivalence while preserving the normal form of $\tilde{C}$. Details are omitted. ☐

To compute $\tilde{\alpha}$, one substitutes the normal forms of Proposition 8 in the determining system (14)eq, (15).

Remark 9. The method using the system (14), (15) does not guarantee non-triviality of results. For instance, in the nilpotent case one always obtains a trivial solution

$$\begin{pmatrix} 0 & a_{12} \\ -a_{21} & 0 \end{pmatrix} dx + \begin{pmatrix} 0 & b_{12} \\ -b_{21} & 0 \end{pmatrix} dy$$

which coincides with

$$\bar{d}Q - [\alpha, Q] \quad \text{with} \quad Q = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},$$

and should be always excluded from the solution set.

5. Example

Example 10. For the mKdV equation $u_t + u_{xxx} - 6u^2 u_x = 0$, the ZCR $\alpha = A(\lambda)\,dx + B(\lambda)\,dt$ was found in [2]; we write it here in the normal form

$$\begin{aligned} A(\lambda) &= \begin{pmatrix} u & \lambda \\ 1 & -u \end{pmatrix}, \\ B(\lambda) &= \begin{pmatrix} -u_{xx} + 2u^3 - 4\lambda u & 2\lambda u_x + 2\lambda u^2 - 4\lambda^2 \\ -2u_x + 2u^2 - 4\lambda & u_{xx} - 2u^3 + 4\lambda u \end{pmatrix}. \end{aligned}$$

Then $A_t - B_x + [A, B] = (u_t + u_{xxx} - 6u^2 u_x) \cdot C$ with $C = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and hence C is the characteristic matrix for this ZCR by formula (5). Let us compute $\bar{H}^1_\alpha$ for $\lambda = 1$, i.e., with

$$\begin{aligned} A = A(1) &= \begin{pmatrix} u & 1 \\ 1 & -u \end{pmatrix}, \\ B = B(1) &= \begin{pmatrix} -u_{xx} + 2u^3 - 4u & 2u_x + 2u^2 - 4 \\ -2u_x + 2u^2 - 4 & u_{xx} - 2u^3 + 4u \end{pmatrix}. \end{aligned}$$

As C, A are already in normal form (17), the next step is to write conditions (2) and (4) for $\tilde{A}$, $\tilde{C}$ in normal form (19). We obtain the following system of six linear equations in total derivatives for the six unknowns a_{31}, a_{32}, b_{31}, b_{32}, b_{41}, s:

$$\begin{aligned} 0 &= -D_t a_{31} + 2(u_x - u^2 + 2)a_{32} + D_x b_{31} + b_{32} - b_{41}, \\ 0 &= -4(u_x + u^2 - 2)a_{31} - (D_t + 2u_{xx} - 4u^3 + 8u)a_{32} + 2b_{31} \\ &\qquad + (D_x - 2u)b_{32}, \\ 0 &= 4(u^2 - u_x - 2)a_{31} - 2b_{31} + (D_x + 2u)b_{41}, \\ &\qquad - [D_t + D_{xxx} + 6(u^2 - 2)D_x]s - 6D_x a_{32}, \\ 0 &= -6(D_{xx} - 2uD_x)s + 4(D_x - 4u)a_{31} \\ &\qquad - 2(D_{xx} - 4uD_x - 2u_x - 2u^2 + 8)a_{32} - 2b_{32}, \\ 0 &= 6(D_{xx} + 2uD_x)s + 4(D_x + 4u)a_{31} + 8a_{32} + 2b_{41}. \end{aligned}$$

Solution vectors depending on second-order jet coordinates at most are multiples of $(a_{31}, a_{32}, b_{31}, b_{32}, b_{41}) = (0, \frac{1}{2}, -2u, u_x + u^2 - 4, -2)$, while $s = \text{const}$. This solution obviously satisfies $A' = \partial A/\partial\lambda|_{\lambda=1}$, $B' = \partial B/\partial\lambda|_{\lambda=1}$, hence corresponds to the nonremovable parameter λ in α.

Appendix

Let us demonstrate that the series of nonlocal conservation laws of the Burgers equation as generated in [21, Ch. I, Example 7.10a] is actually finite. First we recall briefly the procedure [21, Ch. I, Prop. 7.5].

For any ZCR

$$\begin{pmatrix} \frac{1}{2}a_2 & \frac{1}{2}a_1 - \frac{1}{2}a_3 \\ \frac{1}{2}a_1 + \frac{1}{2}a_3 & -\frac{1}{2}a_2 \end{pmatrix} dx + \begin{pmatrix} \frac{1}{2}b_2 & \frac{1}{2}b_1 - \frac{1}{2}b_3 \\ \frac{1}{2}b_1 + \frac{1}{2}b_3 & -\frac{1}{2}b_2 \end{pmatrix} dt,$$

the equations

$$\begin{aligned} \phi_x &= a_1 \sin\phi + a_2 \cos\phi + a_3, \\ \phi_t &= b_1 \sin\phi + b_2 \cos\phi + b_3 \end{aligned} \tag{20}$$

are compatible and the 1-form

$$\omega = (a_1 \cos\phi - a_2 \sin\phi)\, dx + (b_1 \cos\phi - b_2 \sin\phi)\, dt$$

is $\bar{d}$-closed and hence represents a nonlocal conservation law. Considering a_i, b_i to be analytic functions of the parameter η, one may expand ϕ into a power series $\phi = \sum \phi_i \eta^i$; and likewise $\omega = \sum \omega_i \eta^i$. Coefficients ω_i then form an infinite series of conservation laws, although trivial if the parameter η is removable.

For the Burgers equation we have $a_1 = \frac{1}{2}u$, $a_2 = \eta$, $a_3 = -\eta$, $b_1 = \frac{1}{2}u_x + \frac{1}{4}u^2$, $b_2 = \frac{1}{2}\eta u$, $b_3 = -\frac{1}{2}\eta u$. Upon introducing two nonlocal variables $v = \int u\, dx$, $z = \int \mathrm{e}^{v/2}\, dx$ equations (20) on ϕ are easily solved as

$$\phi = 2 \arctan \frac{\mathrm{e}^{v/2}}{\eta z + 1}.$$

The variables v, z are potentials of two nontrivial and independent conservation laws

$$\gamma_1 = u\, dx + (u_x + \tfrac{1}{2}u^2)\, dt, \qquad \gamma_2 = \mathrm{e}^{v/2}(dx + \tfrac{1}{2}\, dt).$$

The Taylor expansion of ϕ around $\eta = 0$ is $\phi = \sum_i \phi_i \eta^i = 2 \arctan \mathrm{e}^{v/2} - 2\mathrm{e}^{v/2}(\mathrm{e}^v + 1)^{-1} z\eta + 4\mathrm{e}^{v/2}(\mathrm{e}^v + 1)^{-2} z^2 \eta^2 + 4\mathrm{e}^{v/2}(\mathrm{e}^v - 3)(\mathrm{e}^v + 1)^{-3} z^3 \eta^3 + \cdots$. Coefficients ϕ_i coincide with those found in [21] modulo integration constants. But one readily observes that ω is a coboundary, namely $\omega = \bar{d}\mu$ for

$$\mu = \tfrac{1}{2}v - \ln(\eta^2 z^2 + 2\eta z + 1 + \mathrm{e}^v).$$

Since ω is exact, so is each of its coefficients: $\omega_i = \bar{d}\mu_i$, where $\mu = \sum_i \mu_i \eta^i$. Thus, the only nontrivial conservation laws of the Burgers equation that are generated by the procedure are the two γ's above. In particular, the potential h introduced in [21] is related to v as $h = \frac{1}{2}v - \ln(\mathrm{e}^v + 1)$.

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