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Alexander Verbovetsky

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The Diffiety Institute
Polevaya 6-45, Pereslavl-Zalessky, 152140 Russia.

**Remarks on two approaches to the horizontal cohomology:
compatibility complex and the Koszul-Tate resolution**

ALEXANDER VERBOVETSKY

ABSTRACT. The Koszul-Tate resolution is described in the context of the geometry of jet spaces and differential equations. The application due to Barnich, Brandt, and Henneaux of this resolution to computing the horizontal cohomology is analyzed. Relations with the Vinogradov spectral sequence are discussed.

The Perl motto is “There’s more than one way to do it.” Divining how many more is left as an exercise to the reader.

Larry Wall, *The Perl man page*

1. INTRODUCTION

This paper is concerned with general methods for computing horizontal (also called “characteristic”) cohomology of systems of nonlinear partial differential equations. There are at least two such methods. One stems from the fact that the horizontal cohomology is the column $E_1^{0,\bullet}$ of the Vinogradov spectral sequence [8, 9, 10] and thereby related to the terms $E_1^{p,\bullet}$ for $p > 0$. These terms can be computed via the compatibility complex for the linearization of the system under consideration. For a detailed description of this technique the reader should consult [6, 7].

The second method was proposed in [1] and is based on the Koszul-Tate resolution [4]. The purpose of the present paper is to describe this method in the language of the geometry of differential equations (see, e.g., [2, 5]) and look into relationships between this method and the former one.

We restrict the discussion to a general theory. As an example we refer to the p -form gauge theory that was explicitly worked out in [3] by means of the Koszul-Tate resolution and in [7, 6] by means of the Vinogradov spectral sequence.

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2. ON COMPATIBILITY OPERATORS

We assume that the reader is familiar with the geometry of jet spaces and differential equations, including the horizontal cohomology, the Vinogradov spectral sequence, the compatibility complex, and the k -line theorem. This material can be found in [6, 7].

Let $\pi: E \rightarrow M$ be a vector bundle and $\pi_\infty: J^\infty(\pi) \rightarrow M$ the associated infinite jet bundle. The standard coordinates on the space $J^\infty(\pi)$ are coordinates x_i of the manifold M and infinitely many coordinates u_σ^j , corresponding to partial derivatives of sections $u = (u^1, \dots, u^m)$ of the bundle π .

Remark. All constructions of the present paper can be readily generalized to the case of vector bundle π with super fibers.

Let $P_1 = \Gamma(\pi_\infty^*(\alpha_1))$ be a horizontal module on $J^\infty(\pi)$ (as usual, a module is always the module of sections of a vector bundle; Γ denotes the functor that takes bundles to the corresponding modules), $\mathcal{E} = \{F = 0\} \subset J^k(\pi)$ a formally integrable equation, determined by a section $F \in P_1$, and $\mathcal{E}^\infty \subset J^\infty(\pi)$ its infinite prolongation.

Remark. Physicists called coordinates along fibers of α_1 *antifields* and say that they have the *antighost number* 1 (hence the notation P_1).

Regularity Condition ([4]). We shall assume that there exists an open submanifold $\mathcal{U} \subset J^\infty(\pi)$ and an isomorphism $\nu: \mathcal{U} \rightarrow \mathcal{E}^\infty \times \mathcal{V}$, where \mathcal{V} is a star-shaped neighborhood of the zero in \mathbb{R}^∞ , such that $\mathcal{U} \supset \mathcal{E}^\infty$, $\nu(\theta) = (\theta, 0)$ for any point $\theta \in \mathcal{E}^\infty$, and the composition $v: \mathcal{U} \xrightarrow{\nu} \mathcal{E}^\infty \times \mathcal{V} \rightarrow \mathcal{V}$ of the isomorphism ν and the projection on the second factor has the form

$$v = (v_1, v_2, v_3, \dots, v_s, \dots),$$

where $v_s = D_{\sigma_s}(F_{i_s})$.

Because of the regularity condition, a function f on \mathcal{U} vanishes on \mathcal{E}^∞ if and only if it has the form $f = \Delta(F)$ for some operator $\Delta \in \mathcal{CDiff}(P_1, \mathcal{F})$, where \mathcal{F} is the algebra of functions on \mathcal{U} .

From here on, we fix a manifold \mathcal{U} and will consider it instead of the whole jet space $J^\infty(\pi)$.

Lemma 1. *Let $Q = \Gamma(\pi_\infty^*(\alpha))$ be a horizontal module; then the kernel of the map*

$$Q \rightarrow \mathcal{CDiff}(\mathcal{K}|_{\mathcal{E}^\infty}, Q|_{\mathcal{E}^\infty}), \quad q \mapsto \ell_q|_{\mathcal{E}^\infty},$$

where $\mathcal{K} = \Gamma(\pi_\infty^*(\pi))$ and ℓ_q is the linearization of q , has the form $\mathcal{I}^2 Q \oplus Q' \subset Q$, where $\mathcal{I} = \{\Delta(F) \mid \Delta \in \mathcal{CDiff}(P_1, \mathcal{F})\} \subset \mathcal{F}$ is the ideal of the equation \mathcal{E}^∞ and $Q' = \{\pi_\infty^*(s) \mid s \in \Gamma(\alpha)\}$. (In other words, the set $\mathcal{I}^2 Q \subset Q$ consists of elements of the form $\Delta(F, F)$ for $\Delta \in \mathcal{CDiff}(P_1, \mathcal{CDiff}(P_1, Q))$.)

Proof. It suffices to consider the case $Q = \mathcal{F}$ and to prove the statement in a local chart. Take a function $f \in \mathcal{F}$ such that $\ell_f = 0$. Then $\partial f / \partial u_\sigma^j \in \mathcal{I}$. Now let us change coordinates to (x_i, v_s, w_t) , where v_s are the coordinates along \mathcal{V}

from the regularity condition and w_t are arbitrary coordinates along \mathcal{E}^∞ . We have $\partial f/\partial v_s \in \mathcal{I}$ and $\partial f/\partial w_t \in \mathcal{I}$. Write the function f in the form

$$f = f_0(x_i, w_t) + \sum_s f_s(x_i, w_t)v_s + f',$$

where $f' \in \mathcal{I}^2$. Then $\partial f/\partial v_s = f_s + \partial f'/\partial v_s \in \mathcal{I}$, hence $f_s = 0$, and $\partial f/\partial w_t = \partial f_0/\partial w_t + \partial f'/\partial w_t \in \mathcal{I}$, so that $\partial f/\partial w_t = 0$. Thus, $f \in \mathcal{I}^2 \oplus \pi_\infty^*(C^\infty(M))$. \square

Proposition 2. *An operator $\Delta \in \mathcal{CDiff}(P_1|_{\mathcal{E}^\infty}, P_2|_{\mathcal{E}^\infty})$ is the compatibility operator for $\ell_F|_{\mathcal{E}^\infty}$ if and only if for each operator $\nabla: P_1 \rightarrow Q$ such that $\nabla(F) = 0$ we have $\nabla|_{\mathcal{E}^\infty} = \square \circ \Delta$ for an operator $\square \in \mathcal{CDiff}(P_2|_{\mathcal{E}^\infty}, Q|_{\mathcal{E}^\infty})$.*

Proof. Suppose that $\Delta \in \mathcal{CDiff}(P_1|_{\mathcal{E}^\infty}, P_2|_{\mathcal{E}^\infty})$ is the compatibility operator for $\ell_F|_{\mathcal{E}^\infty}$. Linearizing the equality $\nabla(F) = 0$, we get $\nabla|_{\mathcal{E}^\infty} \circ \ell_F|_{\mathcal{E}^\infty} = 0$. Hence, $\nabla|_{\mathcal{E}^\infty} = \square \circ \Delta$.

Conversely, consider an operator $\nabla \in \mathcal{CDiff}(P_1, Q)$ satisfying the condition $\nabla|_{\mathcal{E}^\infty} \circ \ell_F|_{\mathcal{E}^\infty} = 0$. Since $\ell_{\nabla(F)}|_{\mathcal{E}^\infty} = \nabla|_{\mathcal{E}^\infty} \circ \ell_F|_{\mathcal{E}^\infty} = 0$, we see that $\nabla(F) = \nabla'(F, F)$. Define the operator $\nabla_1 \in \mathcal{CDiff}(P_1, Q)$ by the equality $\nabla_1(p) = \nabla(p) - \nabla'(F, p)$. We have $\nabla_1(F) = 0$, so that $\nabla|_{\mathcal{E}^\infty} = \nabla_1|_{\mathcal{E}^\infty} = \square \circ \Delta$. This completes the proof. \square

An equation is called *normal* if the compatibility operator for the operator $\ell_F|_{\mathcal{E}^\infty}$ is trivial.

Corollary 3. *The equation \mathcal{E}^∞ is normal if and only if for each operator $\nabla: P_1 \rightarrow Q$ the equality $\nabla(F) = 0$ implies $\nabla|_{\mathcal{E}^\infty} = 0$.*

Corollary 4 (from the proof). *Suppose that $\Delta \in \mathcal{CDiff}(P_1|_{\mathcal{E}^\infty}, P_2|_{\mathcal{E}^\infty})$ is the compatibility operator for $\ell_F|_{\mathcal{E}^\infty}$; then there exist an extension $\tilde{\Delta} \in \mathcal{CDiff}(P_1, P_2)$ of Δ such that $\tilde{\Delta}(F) = 0$.*

3. THE KOSZUL-TATE RESOLUTION

Due to the last Corollary there is an operator $\Delta_1: P_1 \rightarrow P_2$ such that $\Delta_1|_{\mathcal{E}^\infty}$ is the compatibility operator for $\ell_F|_{\mathcal{E}^\infty}$ and $\Delta_1(F) = 0$. Pick up such an operator and construct the compatibility complex for it:

$$P_1 \xrightarrow{\Delta_1} P_2 \xrightarrow{\Delta_2} \dots \xrightarrow{\Delta_{k-2}} P_{k-1} \rightarrow 0,$$

where $P_i = \Gamma(\alpha_i)$ for some vector bundles $\alpha_i: V_i \rightarrow \mathcal{U}$.

Remark. Such a complex may not exist (modules P_i may not be projective). In [4] the assumption that it exists is termed as ‘‘off-shell reducibility.’’ The situation then this complex exists only on \mathcal{E}^∞ is called ‘‘on-shell reducibility.’’ So, we here require the off-shell reducibility.

Take the direct sum $\alpha = \bigoplus_{i \geq 1} (\alpha_{2i-1}^{\Pi} \oplus \alpha_{2i})$, where α_{2i-1}^{Π} means the bundle α_{2i-1} with reversed parity of fibers, and consider the supermanifold of horizontal jets of its sections $\bar{J}^{\infty}(\alpha)$.

Each Cartan field on \mathcal{U} can be naturally lifted to $\bar{J}^{\infty}(\alpha)$; these liftings span the Cartan distribution on the horizontal jets $\bar{J}^{\infty}(\alpha)$. It is not hard to check that all the theory of jet spaces can be carried over to the space $\bar{J}^{\infty}(\alpha)$. In coordinate language, the horizontal jet space is a jet space equipped with extra base coordinates u_{σ}^j (so that all functions depend on them as on parameters), with the total derivatives $D_i = \partial/\partial x_i + \sum_{j,\sigma} u_{\sigma}^j \partial/\partial u_{\sigma}^j$ in place of the partial derivatives $\partial/\partial x_i$.

Now let us pull the element F and all operators Δ_i back to $\bar{J}^{\infty}(\alpha)$. We shall treat them as elements of the module $\varkappa(\alpha) = \Gamma(\alpha^*(\alpha))$. Consider the element $\Phi = F + \Delta_1 + \dots + \Delta_{k-2} \in \varkappa(\alpha)$.

Proposition 5. *The odd evolutionary vector field $\delta = \mathfrak{D}_{\Phi}$ is a differential: $\delta^2 = 0$.*

Proof. Since δ is a vector field and $\delta|_{\mathcal{F}} = 0$, it is sufficient to evaluate δ^2 on functions linear along the fibers of the natural projection $\alpha_{\infty}: \bar{J}^{\infty}(\alpha) \rightarrow \mathcal{U}$. Such functions can be naturally identified with \mathcal{C} -differential operators belonging to $\mathcal{C}\text{Diff}(P, \mathcal{F})$, where $P = \Gamma(\alpha) = \bigoplus_{i \geq 1} (\Gamma(\alpha_{2i-1}^{\Pi}) \oplus \Gamma(\alpha_{2i}))$ is a graded \mathcal{F} -module. Define an odd \mathcal{C} -differential operator $\Delta: P \rightarrow P$ by the formula

$$\Delta(p_1, \dots, p_{k-2}, p_{k-1}) = (0, \Delta_1(p_1), \dots, \Delta_{k-2}(p_{k-2})),$$

where $p_i \in P_i = \Gamma(\alpha_i)$. Trivially, $\Delta^2 = 0$ and $\Delta(F) = 0$. It is easy to see that if $\nabla \in \mathcal{C}\text{Diff}(P, \mathcal{F})$ then $\delta(\nabla) = \nabla(F) + \nabla \circ \Delta$, so that $\delta^2(\nabla) = \nabla \circ \Delta^2 = 0$. \square

Denote by $\mathcal{F}^{\text{pol}}(\alpha)$ the subalgebra of the algebra $\mathcal{F}(\alpha)$ of functions on $\bar{J}^{\infty}(\alpha)$ that consists of functions polynomial along the fibers of the projection $\alpha_{\infty}: \bar{J}^{\infty}(\alpha) \rightarrow \mathcal{U}$. We supply the algebra $\mathcal{F}^{\text{pol}}(\alpha)$ with a \mathbb{Z} -grading, $\mathcal{F}^{\text{pol}}(\alpha) = \bigoplus_{i \geq 0} \mathcal{F}_i^{\text{pol}}(\alpha)$, called the *antighost number*, such that fiberwise linear functions on $\bar{J}^{\infty}(\alpha_i)$ have antighost number i and functions on \mathcal{U} have antighost number zero. (Thus the parity of a function is equal to its antighost number modulo 2.) The differential δ reduces the antighost number by 1 and so we have the complex

$$0 \leftarrow \mathcal{F} \xleftarrow{\delta} \mathcal{F}_1^{\text{pol}}(\alpha) \xleftarrow{\delta} \mathcal{F}_2^{\text{pol}}(\alpha) \xleftarrow{\delta} \dots$$

called the *Koszul-Tate complex* [4].

Theorem 6. *The Koszul-Tate complex is a resolution, with the zero homology isomorphic to the algebra $\mathcal{F}(\mathcal{E})$ of functions on the equations \mathcal{E}^{∞} . The homology of the differential group $(\mathcal{F}(\alpha), \delta)$ is equal to $\mathcal{F}(\mathcal{E})$ as well.*

Proof. See [4]. \square

Let us consider the bicomplex $(\mathcal{F}_p^{\text{pol}}(\alpha) \otimes \bar{\Lambda}^q, \delta, (-1)^q \bar{d})$, where $\bar{\Lambda}^q$ is the module of horizontal q -forms on \mathcal{U} . The horizontal differential \bar{d} is well-defined, since there is a natural inclusion $\mathcal{F}_p^{\text{pol}}(\alpha) \otimes \bar{\Lambda}^q \subset \bar{\Lambda}^q(\alpha)$, where $\bar{\Lambda}^q(\alpha)$ is the module of horizontal q -forms on $\bar{J}^\infty(\alpha)$. Using this bicomplex and Theorem 6, we immediately conclude that for $0 \leq q \leq n-1$ the group $\bar{H}^q(\mathcal{E})/\bar{H}^q(\mathcal{U})$ is isomorphic to the $(n-q)^{\text{th}}$ homology group of the complex

$$\bar{H}_0^n(\alpha) \xleftarrow{\delta} \bar{H}_1^n(\alpha) \xleftarrow{\delta} \bar{H}_2^n(\alpha) \xleftarrow{\delta} \dots, \quad (1)$$

where $\bar{H}_i^n(\alpha) = H^n(\mathcal{F}_i^{\text{pol}}(\alpha) \otimes \bar{\Lambda}^q, \bar{d})$. Application of the one-line theorem for \mathcal{U} yields the following reformulation of this result.

Theorem 7. *For $0 \leq q \leq n-1$ there is the isomorphism*

$$\bar{H}^q(\mathcal{E})/H^q(M) = H_{n-q}(\bar{H}_\bullet^n(\alpha), \delta).$$

Our main concern now is to compute the homology of complex (1). To this end it is convenient to drop the zero-term $\bar{H}_0^n(\alpha)$ and to embed the rest of the complex to the complex $(\hat{\mathcal{Z}}(\alpha), \delta + \ell_\Phi^*)$ by means of the Euler operator

$$\bigoplus_{i \geq 1} \bar{H}_i^n(\alpha) \rightarrow \hat{\mathcal{Z}}(\alpha) \quad (2)$$

on $\bar{J}^\infty(\alpha)$.

Remark. We use here two facts:

- (1) the map $\delta + \ell_\Phi^*: \hat{\mathcal{Z}}(\alpha) \rightarrow \hat{\mathcal{Z}}(\alpha)$ is a differential: $(\delta + \ell_\Phi^*)^2 = 0$;
- (2) the Euler operator (2) is a chain map.

Both of these statements follow from the infinitesimal Stokes formula in the term E_1 of the Vinogradov spectral sequence on $\bar{J}^\infty(\alpha)$ (see [10, Sec. 9.10]¹).

Proposition 8. *The operator $\ell_\Phi^*: \hat{\mathcal{Z}}(\alpha) \rightarrow \hat{\mathcal{Z}}(\alpha)$ is a differential, i.e., $(\ell_\Phi^*)^2 = 0$.*

Proof. Pick up an element $\theta \in \hat{\mathcal{Z}}(\alpha)$. It can be thought of as a nonlinear operator $\theta: P \rightarrow \hat{P}$. Obviously, $\ell_\Phi^*(\theta) = (-1)^{p(\theta)} \Delta^* \circ \theta$, where $p(\theta)$ is the parity of θ . Hence, $(\ell_\Phi^*)^2(\theta) = -(\Delta^*)^2 \circ \theta = 0$. \square

Remark. Note that the operator $\Delta^*: \hat{P} \rightarrow \hat{P}$ has the form

$$\Delta^*(\hat{p}_1, \hat{p}_2, \dots, \hat{p}_{k-1}) = (-\Delta_1^*(\hat{p}_2), \Delta_2^*(\hat{p}_3), \dots, (-1)^{k-2} \Delta_{k-2}^*(\hat{p}_{k-1})),$$

where $\hat{p}_i \in \hat{P}_i$.

¹Note a misprint in [10, eq. (9.10.5)]. The correct formula (in the notation of [10]) is

$$\chi\{\varphi\} = l_\varphi(\chi) + l_\chi^*(\varphi) = \mathfrak{Z}_\chi(\varphi) + l_\chi^*(\varphi).$$

Thus, we see that the differentials δ and ℓ_{Φ}^* define a bicomplex structure in $\hat{\mathcal{Z}}(\alpha)$. The same holds true for the polynomial part $\hat{\mathcal{Z}}^{\text{pol}}(\alpha) = \mathcal{F}^{\text{pol}}(\alpha) \otimes \hat{P}$. This enables an easy computation of the homology of the total complex $(\hat{\mathcal{Z}}(\alpha), \delta + \ell_{\Phi}^*)$ and its polynomial subcomplex.

Proposition 9. *The homologies of complexes*

$$(\hat{\mathcal{Z}}(\alpha), \delta + \ell_{\Phi}^*) \quad \text{and} \quad (\hat{\mathcal{Z}}^{\text{pol}}(\alpha), \delta + \ell_{\Phi}^*)$$

coincide and equal the homology of the complex

$$0 \leftarrow \hat{P}_1 \xleftarrow{\Delta_1^*} \hat{P}_2 \xleftarrow{\Delta_2^*} \dots \xleftarrow{\Delta_{k-2}^*} \hat{P}_{k-1} \leftarrow 0 \quad \text{on the equation } \mathcal{E}^\infty. \quad (3)$$

Corollary 10. *Each cycle ψ belonging to $\hat{\mathcal{Z}}_+^{\text{pol}}(\alpha) = (\bigoplus_{i \geq 1}^\infty \mathcal{F}_i^{\text{pol}}(\alpha)) \otimes \hat{P}$ is a boundary.*

From the one-line theorem for $\bar{J}^\infty(\alpha)$ it follows that the image of the mapping (2) coincides with the complex $(\hat{\mathcal{Z}}^{\text{SP}}(\alpha), \delta + \ell_{\Phi}^*)$, where

$$\hat{\mathcal{Z}}^{\text{SP}}(\alpha) = \{ \psi \in \hat{\mathcal{Z}}^{\text{pol}}(\alpha) \mid \ell_\psi^* = \ell_\psi \}.$$

Remark. It should be noted that $\hat{\mathcal{Z}}^{\text{SP}}(\alpha)$ does not inherit the bicomplex structure of $\hat{\mathcal{Z}}^{\text{pol}}(\alpha)$.

The space $\hat{\mathcal{Z}}^{\text{SP}}(\alpha)$ can be expanded in the sum

$$\hat{\mathcal{Z}}^{\text{SP}}(\alpha) = \hat{P} \oplus \hat{\mathcal{Z}}_+^{\text{SP}}(\alpha),$$

where $\hat{\mathcal{Z}}_+^{\text{SP}}(\alpha) = \hat{\mathcal{Z}}^{\text{SP}}(\alpha) \cap \hat{\mathcal{Z}}_+^{\text{pol}}(\alpha)$.

Proposition 11. *Each cycle ψ belonging to $\hat{\mathcal{Z}}_+^{\text{SP}}(\alpha)$ is a boundary in the complex $(\hat{\mathcal{Z}}^{\text{SP}}(\alpha), \delta + \ell_{\Phi}^*)$.*

Proof. We prove the statement by identifying in a natural way elements $\psi \in \hat{\mathcal{Z}}^{\text{pol}}(\alpha)$ with graded symmetric multilinear \mathcal{C} -differential operators

$$\nabla_\psi: P \times \dots \times P \rightarrow \hat{P}.$$

It is not hard to check that $\psi \in \hat{\mathcal{Z}}^{\text{SP}}(\alpha)$ if and only if the corresponding operator ∇_ψ is selfadjoint: $\nabla_\psi = \nabla_\psi^*$ (since ∇_ψ is symmetric, self-adjointness in one argument implies self-adjointness in the other arguments). Consider the projector $S: \hat{\mathcal{Z}}^{\text{pol}}(\alpha) \rightarrow \hat{\mathcal{Z}}^{\text{SP}}(\alpha)$ given by $S(\nabla) = (\nabla + \nabla^*)/2$. Obviously, $S^2 = S$ and $S \circ (\delta + \ell_{\Phi}^*) = (\delta + \ell_{\Phi}^*) \circ S$. Now, if a cycle ∇ belongs to $\hat{\mathcal{Z}}_+^{\text{SP}}(\alpha)$ then by Corollary 10 we have $\nabla = (\delta + \ell_{\Phi}^*)\nabla'$. This gives $\nabla = S(\nabla) = S(\delta + \ell_{\Phi}^*)\nabla' = (\delta + \ell_{\Phi}^*)(S(\nabla'))$, which is the desired conclusion. \square

Corollary 12. *Complex (1) is exact in terms $\bar{H}_i^n(\alpha)$ for $i \geq k$.*

Corollary 13. *If $0 \leq q \leq n - k$ then $\bar{H}^q(\mathcal{E}) = H^q(M)$.*

Take a cycle $\psi \in \hat{\mathcal{Z}}^{\text{sp}}(\alpha)$ and write it in the form $\psi = \psi_0 + \psi_1$, where $\psi_0 \in \hat{P}$ and $\psi_1 \in \hat{\mathcal{Z}}_+^{\text{sp}}(\alpha)$. Obviously, $\psi_0|_{\mathcal{E}^\infty}$ is a cycle of complex (3). So, we have constructed a map from homology of complex (1) to homology of complex (3).

Let us examine this map more fully in dimension 1. We have $\bar{H}_1^n(\alpha) = \hat{P}_1$ and the map being considered is just the restriction $\hat{P}_1 \rightarrow \hat{P}_1|_{\mathcal{E}^\infty}$. Further, an element $\beta \in \bar{H}_1^n(\alpha) = \hat{P}_1$ is a cycle if and only if the horizontal cohomology class of the n -form $\langle \beta, F \rangle$ is trivial, so that $0 = \ell_{\langle \beta, F \rangle}^*(1) = \ell_\beta^*(F) + \ell_F^*(\beta)$ and, hence, $\ell_F^*|_{\mathcal{E}^\infty}(\beta|_{\mathcal{E}^\infty}) = 0$. Thus, we get a map from homology of complex (1) to homology of the complex

$$\varkappa \xleftarrow{\ell_F^*} \hat{P}_1 \xleftarrow{\Delta_1^*} \hat{P}_2 \xleftarrow{\Delta_2^*} \dots \xleftarrow{\Delta_{k-2}^*} \hat{P}_{k-1} \leftarrow 0 \quad \text{on the equation } \mathcal{E}^\infty. \quad (4)$$

The application of Theorem 7 yields the map

$$\bar{H}^q(\mathcal{E}) \rightarrow H_{n-q}(\hat{P}_\bullet, \Delta_\bullet^*). \quad (5)$$

It is straightforward to check that this map coincides with the differential $d_1^{0,q}: E_1^{0,q} = \bar{H}^q(\mathcal{E}) \rightarrow E_1^{1,q} = H_{n-q}(\hat{P}_\bullet, \Delta_\bullet^*)$ of the Vinogradov spectral sequence on \mathcal{E}^∞ .

4. A COMPARISON

As it is seen from the last paragraphs of the previous section, the homology of complex (4) is essential to computing the horizontal cohomology via the Koszul-Tate resolution, similarly to what happens when using the Vinogradov spectral sequence. This bridges the gap between two approaches. Otherwise they are diverged considerably.

As an example, let us discuss the computation of conservation laws for a normal equation (i.e., such that $\Delta_1 = 0$). The application of the Koszul-Tate resolution gives the following result.

Proposition 14. *Let \mathcal{E}^∞ be a normal equation. The space of conservation laws of \mathcal{E}^∞ is isomorphic to the space of solutions of the equation*

$$\ell_F^*(\psi) + \ell_\psi^*(F) = 0,$$

where $\psi \in \hat{P}_1/\Theta$ and $\Theta = \{ \square(F) \in \hat{P}_1 \mid \square \in \mathcal{C}\text{Diff}(P_1, \hat{P}_1), \square = -\square^* \}$.

Another result is obtained by means of the Vinogradov spectral sequence.

Proposition 15. *Let \mathcal{E}^∞ be a normal equation. The space of conservation laws of \mathcal{E}^∞ is a subset of the space of solutions of the equation*

$$\ell_F^*(\psi) = 0 \quad \text{on } \mathcal{E}^\infty, \quad (6)$$

where $\psi \in \hat{P}_1|_{\mathcal{E}^\infty}$. A solution ψ corresponds to a conservation law if and only if on \mathcal{E}^∞ there exists a selfadjoint \mathcal{C} -differential operator $\nabla: P_1 \rightarrow \hat{P}_1$ such that

$$\ell_\psi + \Delta^* = \nabla \circ \ell_F, \quad \nabla^* = \nabla \quad \text{on } \mathcal{E}^\infty,$$

where $\Delta: P_1 \rightarrow \hat{\mathcal{Z}}$ is a \mathcal{C} -differential operator satisfying on \mathcal{U} the equality $\ell_F^*(\psi) = \Delta(F)$.

Thus, both of these Propositions say that to compute conservation laws of a normal equation we should start with solving equation (6). Proposition 15 also says that if ψ vanishes on \mathcal{E}^∞ then the corresponding conservation law is trivial (essentially, this is the basic content of the Proposition). Proposition 14 implies a weaker result: it guarantees triviality of conservation laws that correspond to elements ψ of the form $\psi = \square(F)$ for skew-adjoint operators \square only.

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INDEPENDENT UNIVERSITY OF MOSCOW. CORRESPONDENCE TO:

A. VERBOVETSKY, PROFSOYUZNAYA 98-9-132, 117485 MOSCOW, RUSSIA.

E-mail address: verbovet@mccme.ru