

**An example of  $(3 + 1)$ -dimensional  
integrable system**

by

V. V. TRUSHKOV

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**The Diffiety Institute**  
Polevaya 6-45, Pereslavl-Zalessky, 152140 Russia.

# An example of (3 + 1)-dimensional integrable system

V. V. TRUSHKOV

ABSTRACT. An example of a (3+1)-dimensional integrable system is considered, infinite series of divergent forms are described. Classical symmetries for this system and self-similar exact solutions are found.

## 1. INTRODUCTION

A three-dimensional hydrodynamic model of the formation of sedimentary basins is considered. This model is described by quasi-steady equations of motion of incompressible fluid and by evolutionary equations of the density transfer:

$$(1) \quad \begin{aligned} \nabla p &= \mu \Delta \vec{v} + g\rho\delta_{i3}, \\ \operatorname{div} \vec{v} &= 0, \\ \rho_t + \operatorname{div}(\rho\vec{v}) &= 0, \end{aligned}$$

where  $\rho_t = \partial\rho/\partial t$ .

In Section 2 of the paper theoretical background is introduced. In Section 3 symmetries of system (1) are described. In Section 5 conservation laws are described and integrability of system (1) is shown. In the fifth section some exact self-similar solutions are obtained.

## 2. THEORETICAL BACKGROUND

Consider the space  $J^k(n, m)$  with the coordinates

$$x_1, \dots, x_n, u^1, \dots, u^m, \dots, u_\sigma^1, \dots, u_\sigma^m, \dots,$$

where  $\sigma = (i_1, \dots, i_n)$  and  $|\sigma| \leq k$ . This space is called the *space of jets* of order  $k$ . Then an equation  $\mathcal{E}$  can be understood as a subset in  $J^k(n, m)$  defined by the relations

$$(2) \quad F^1(x, u, \dots, u_\sigma, \dots) = 0, \dots, F^r(x, u, \dots, u_\sigma, \dots) = 0.$$

If all  $F^i$  are smooth functions and the system  $F^1, \dots, F^r$  is of maximal rank, then this subset is a smooth submanifold of codimension  $r$ . Thus, *differential equations are smooth submanifolds in the jet spaces*.

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Let  $f = (f^1(x), \dots, f^m(x))$  be a smooth vector function. Then  $f$  determines an  $n$ -dimensional submanifold in  $J^k(n, m)$  defined by

$$u_\sigma^j = \frac{\partial^{|\sigma|} f^j}{\partial x^\sigma}, \quad j = 1, \dots, m, \quad |\sigma| \leq k,$$

(here and below we formally set  $u_{(0, \dots, 0)} = u$ ). This submanifold is called the  $k$ -jet of  $f$  and is denoted by  $j_k(f)$ . It easily seen that  $f$  satisfies equation (2) if and only if  $j_k(f)$  lies in the corresponding submanifold in  $J^k(n, m)$ . In other words, solutions of  $\mathcal{E}$  are identified with jets lying in the equation manifold. Hence, the following definition is justified.

**Definition 1.** Let  $\mathcal{E} \subset J^k(n, m)$  be a differential equation. A *symmetry* of  $\mathcal{E}$  is a diffeomorphism  $\varphi: J^k(n, m) \rightarrow J^k(n, m)$  such that

- (i)  $\varphi(\mathcal{E}) = \mathcal{E}$  and
- (ii)  $\varphi(j_k(f))$  is of the form  $j_k(f_\varphi)$  for every section  $f$ .

A transformation is called a *Lie transformation*, if it satisfies condition (ii).

**Definition 2.** A vector field  $X$  on the manifold  $J^k(n, m)$  is called a *Lie field*, if the displacements along its trajectories are the Lie transformations.

**Definition 3.** A Lie field is called a *classical infinitesimal symmetry* of the equation  $\mathcal{E} \subset J^k(n, m)$ , if it is tangent to  $\mathcal{E}$ .

We define the operators

$$(3) \quad D_\alpha^{[k]} = \frac{\partial}{\partial x_\alpha} + \sum_{j=1}^m \sum_{|\sigma| \leq k} u_{\sigma+1_\alpha}^j \frac{\partial}{\partial u_\sigma^j}.$$

Then one can easily see that the system

$$(4) \quad F = 0, \quad D_1^{[k]}(F) = 0, \quad \dots, \quad D_n^{[k]}(F) = 0$$

and initial equation (2) have the same solutions. If  $\mathcal{E} \subset J^k(n, m)$  is the submanifold corresponding to (2), then the submanifold (probably, with singularities) in the jet space  $J^{k+1}(n, m)$  corresponding to (4) is denoted by  $\mathcal{E}^1$  and called the *first prolongation* of  $\mathcal{E}$ .

Now, we define by induction  $\mathcal{E}^{l+1} = (\mathcal{E}^l)^1$  and see that the set  $\mathcal{E}^l$  lies in  $J^{k+l}(n, m)$ . We call  $\mathcal{E}^l$  the  $l$ -th *prolongation* of  $\mathcal{E}$ .

Let  $J^\infty(n, m)$  be a space with coordinates

$$x_1, \dots, x_n, u^1, \dots, u^m, \dots, u_\sigma^1, \dots, u_\sigma^m, \dots,$$

where  $|\sigma|$  is unlimited. This space is called the *space of infinite jets* and the whole series of prolongations of the equation  $\mathcal{E}$  determines a submanifold  $\mathcal{E}^\infty$  in  $J^\infty(n, m)$  which is called the *infinite prolongation* of the equation  $\mathcal{E}$ . The infinite prolongations are expressed by relations

$$D_\sigma(F) = 0, \quad |\sigma| \geq 0,$$

where  $D_\sigma = D_1^{i_1} \circ \dots \circ D_n^{i_n}$  and

$$(5) \quad D_\alpha = \frac{\partial}{\partial x_\alpha} + \sum_{j, \sigma} u_{\sigma+1_\alpha}^j \frac{\partial}{\partial u_\sigma^j}$$

are infinite counterparts of (3) and are called *total derivatives*.

Expressions of the form (5) can be considered as vector fields on  $J^\infty(n, m)$ . Thus, at any point  $\theta \in J^\infty(n, m)$  an  $n$ -dimensional plane  $\mathcal{C}_\theta$  arises spanned by vectors of these fields. It means that there exists a distribution  $\mathcal{C}: \theta \mapsto \mathcal{C}_\theta$  on  $J^\infty(n, m)$ . We call it the *Cartan distribution*. Its importance to the theory of differential equations is explained by the following

**Theorem 1** (see [1]). *If  $\mathcal{E}^\infty \subset J^\infty(n, m)$  is the infinite prolongation, then  $\mathcal{C}_\theta$  is tangent to  $\mathcal{E}^\infty$  at any point  $\theta \in \mathcal{E}^\infty$ .*

*An  $n$ -dimensional submanifold in  $\mathcal{E}^\infty$  nondegenerately projecting to the space of independent variables is a solution of  $\mathcal{E}$  if and only if it is a maximal integrable manifold of the Cartan distribution.*

Hence,  $\mathcal{E}^\infty$  carries the structure which completely determines solutions of  $\mathcal{E}$  and this structure is the Cartan distribution.

Consider the sets

$$\begin{aligned} \mathcal{CD}(\mathcal{E}^\infty) &= \{X \in D(\mathcal{E}^\infty) \mid X(\omega) = 0, \forall \omega \in \mathcal{C}\Lambda^1(\mathcal{E}^\infty)\}, \\ D_{\mathcal{C}}(\mathcal{E}^\infty) &= \{X \in D(\mathcal{E}^\infty) \mid [X, \mathcal{CD}(\mathcal{E}^\infty)] \subset \mathcal{CD}(\mathcal{E}^\infty)\}. \end{aligned}$$

Note that the set  $D_{\mathcal{C}}(\mathcal{E}^\infty)$  is a subalgebra in the Lie algebra of vector fields on  $\mathcal{E}^\infty$  and  $\mathcal{CD}(\mathcal{E}^\infty)$  is an ideal in  $D_{\mathcal{C}}(\mathcal{E}^\infty)$ .

The Lie  $\mathbb{R}$ -algebra

$$\text{sym } \mathcal{E} = D_{\mathcal{C}}(\mathcal{E}^\infty) / \mathcal{CD}(\mathcal{E}^\infty)$$

is the Lie algebra of symmetries of the Cartan distribution on  $\mathcal{E}^\infty$ .

**Definition 4.** Elements of the Lie algebra  $\text{sym } \mathcal{E}$  are called higher (infinitesimal) symmetries of the equation  $\mathcal{E}$ .

**Theorem 2** (see [1]). *Let  $\mathcal{E} \subset J^k(n, m)$  be an equation given by the system  $\{F_\alpha = 0\}$  and the functions  $F_\alpha$  be chosen in such a way that the graph of  $F = (\dots, F_\alpha, \dots)$  intersects the space of independent variables transversally, then Lie algebra  $\text{sym } \mathcal{E}$  is isomorphic to the Lie algebra of solutions of the equations*

$$(6) \quad \ell_{F_\alpha}^{\mathcal{E}}(\varphi) = 0, \quad \alpha = 1, \dots, r, \quad \varphi = (\varphi^1, \dots, \varphi^m),$$

where  $\varphi^j$  are smooth functions on  $\mathcal{E}^\infty$ ,

$$\ell_\psi = \sum_{j, \sigma} \frac{\partial \psi}{\partial u_\sigma^j} D_\sigma^{(j)}$$

and  $\ell_\psi^{\mathcal{E}}$  is the restriction of  $\ell_\psi$  to  $\mathcal{E}^\infty$ . This isomorphism is given by the correspondence

$$\varphi \mapsto \mathfrak{D}_\varphi = \sum_{\sigma, j} D_\sigma(\varphi^j) \frac{\partial}{\partial u_\sigma^j}$$

The function  $\varphi$  is called the *generating function* corresponds to a symmetry of equation.

Generating functions of conservation laws correspond to solutions of the equation formally adjoint to (6) (see [1]).

## 3. SYMMETRIES

Consider a system of equations for the generating function of symmetries of equation (1)

$$\begin{aligned}
D_z^2(\varphi_u) &= -\frac{1}{\mu}(D_x^2(\varphi_u) + D_y^2(\varphi_u)) + D_x(\varphi_p), \\
D_z^2(\varphi_v) &= -\frac{1}{\mu}(D_x^2(\varphi_v) + D_y^2(\varphi_v)) + D_y(\varphi_p), \\
D_z(\varphi_p) &= \mu(D_x^2(\varphi_w) + D_y^2(\varphi_w) - D_x D_z(\varphi_u) - D_y D_z(\varphi_v)) + g\varphi_\rho, \\
D_z(\varphi_w) &= -D_x(\varphi_x) - D_y(\varphi_y), \\
D_z(\varphi_\rho) &= -\left(\frac{\rho_x}{\rho_z}D_x(\varphi_\rho) + \frac{\rho_y}{\rho_z}D_y(\varphi_\rho) + \frac{\rho_t}{\rho_z}D_t(\varphi_\rho) + \frac{u}{\rho_z}\varphi_u \right. \\
&\quad \left. + \frac{v}{\rho_z}\varphi_v + \frac{w}{\rho_z}\varphi_w\right).
\end{aligned}$$

Here  $D_t, D_x, D_y, D_z$  are the total derivatives restricted to the system.

We solve this system on the space of functions in the variables  $t, x, y, z, \rho, p, u, v, w, u_t, u_x, u_y, u_z, v_t, v_x, v_y, v_z, w_t, w_x, w_y, p_t, p_x, p_y, \rho_t, \rho_y, u_{xz}, v_{yz}$ .

**Theorem 3** (Classical symmetries). *The system*

$$\begin{aligned}
\frac{\partial p}{\partial x} &= \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \\
\frac{\partial p}{\partial y} &= \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \\
\frac{\partial p}{\partial z} &= \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + g\rho, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\
\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} &= 0
\end{aligned}$$

has the classical symmetries

$$\begin{aligned}
(T) & \quad \partial_t, \\
(X_\alpha) & \quad \alpha(t)\partial_x + \alpha'(t)\partial_u, \\
(Y_\beta) & \quad \beta(t)\partial_y + \beta'(t)\partial_v, \\
(Z_\gamma) & \quad \gamma(t)\partial_z + \gamma'(t)\partial_w, \\
(PR) & \quad \partial_\rho + gz\partial_p, \\
(P_\varepsilon) & \quad \varepsilon(t)\partial_p, \\
(M_1) & \quad t\partial_t - \rho\partial_\rho - p\partial_p - u\partial_u - v\partial_v - w\partial_w, \\
(M_2) & \quad x\partial_x + y\partial_y + z\partial_z - \rho\partial_\rho + u\partial_u + v\partial_v + w\partial_w, \\
(\Theta_\delta) & \quad \delta(t)(y\partial_x - x\partial_y + v\partial_u - u\partial_v) + \delta'(t)(y\partial_u - x\partial_v).
\end{aligned}$$

	$T$	$X_\alpha$	$Y_\beta$	$Z_\gamma$	$P_\varepsilon$	$PR$	$M_1$	$M_2$	$\Theta_\delta$
$T$	0	$X_{\alpha'}$	$Y_{\beta'}$	$Z_{\gamma'}$	$P_{\varepsilon'}$	0	$T$	0	$\Theta_{\delta'}$
$X_\alpha$		0	0	0	0	0	$-X_{t\alpha'}$	$X_\alpha$	$-Y_{\alpha\delta}$
$Y_\beta$			0	0	0	0	$-Y_{t\beta'}$	$Y_\beta$	$X_{\beta\delta}$
$Z_\gamma$				0	0	$P_{g\gamma}$	$-Z_{t\gamma'}$	$Z_\gamma$	0
$P_\varepsilon$					0	0	$-P_{\varepsilon+t\varepsilon'}$	0	0
$PR$						0	$-PR$	$-PR$	0
$M_1$							0	0	$\Theta_{t\delta'}$
$M_2$								0	0
$\Theta_\delta$									0

FIGURE 1. Commutator relations between classical symmetries

The commutators of symmetries are presented on Fig. 1. In addition, one has

$$\begin{aligned}
 [P_{\varepsilon_1}, P_{\varepsilon_2}] &= 0, & [X_{\alpha_1}, X_{\alpha_2}] &= 0, & [Y_{\beta_1}, Y_{\beta_2}] &= 0, \\
 [Z_{\gamma_1}, Z_{\gamma_2}] &= 0, & [\Theta_{\delta_1}, \Theta_{\delta_2}] &= 0.
 \end{aligned}$$

The symmetry  $T$  corresponds to the translation in  $t$ . The symmetries  $X_\alpha, Y_\beta, Z_\gamma$  correspond to generalized translations in  $x, y, z$  respectively. The symmetries  $P_\varepsilon, PR$  are gauges on trivial solution. The symmetries  $M_1, M_2$  are scale symmetries. The symmetry  $\Theta_\delta$  corresponds to generalized rotation.

Let  $X$  be an infinitesimal symmetry of the equation  $\mathcal{E} \subset J^k(n, m)$ ,  $f_0$  be a solution of this equation,  $\Gamma_{f_0}^k = j_k(f_0)(M) \subset \mathcal{E}$ . Let  $\{A_t\}$  be the one-parameter group of the vector field  $X$ . Then the submanifold  $A_t(\Gamma_{f_0}^k) \subset J^k(n, m)$  is integrable for the Cartan distribution: if  $A_t(\Gamma_{f_0}^k)$  and  $\Gamma_{f_0}^k$  are a horizontal with respect to the projection on base, then  $A_t(\Gamma_{f_0}^k) = \Gamma_{f_t}^k$  for some section  $f_t$  and sufficiently small  $t$ . Moreover, we obtain from the definition of symmetries that  $A_t(\Gamma_{f_0}^k) \subset \mathcal{E}$ . The translation of an initial solution  $f$  to a family of solutions  $f_t$  is called *reproduction* of solution the  $f$  under the action of  $X$ . If  $\varphi$  is the generating function of symmetry  $X = X_\varphi$  then  $f_t$  is a solution of the system of equations

$$\frac{\partial f^j}{\partial t} = \varphi^j|_{\Gamma_f^k}(x, t), \quad f^j(x, 0) = f_0^j(x), \quad j = 1, \dots, m.$$

If  $\varphi|_{\Gamma_f^k}(x, t) = 0$  then  $f(x, t) = f(x, 0)$  for all  $t$ , i.e.,  $f_0$  are stationary point for one-parameter group of transformations  $\{A_t\}$  under its action on section. In other words, the manifold  $\Gamma_{f_0}^k$  is invariant under the action of the vector field  $X_\varphi$ .

**Definition 5.** If  $\Gamma_{f_0}^k$  is invariant under the action of the vector field  $X_\varphi$ , then  $f_0$  is called a  $\varphi$ -invariant solution of  $\mathcal{E}$ .

Applying this technique to (1), we obtain the following

**Theorem 4.** Let  $\rho = \rho(t, x, y, z)$ ,  $p = p(t, x, y, z)$ ,  $u = u(t, x, y, z)$ ,  $v = v(t, x, y, z)$ ,  $w = w(t, x, y, z)$  be a solution of system (1). Then vector-functions

$$\begin{aligned}
(T) \quad & \rho = \rho(t + \tau, x, y, z), \quad p = p(t + \tau, x, y, z) \\
& u = u(t + \tau, x, y, z), \quad v = v(t + \tau, x, y, z) \\
& w = w(t + \tau, x, y, z) \\
(X_\alpha) \quad & \rho = \rho(t, x + \tau\alpha(t), y, z), \quad p = p(t, x + \tau\alpha(t), y, z) \\
& u = u(t, x + \tau\alpha(t), y, z) - \tau\alpha'(t), \quad v = v(t, x + \tau\alpha(t), y, z) \\
& w = w(t, x + \tau\alpha(t), y, z) \\
(Y_\beta) \quad & \rho = \rho(t, x, y + \tau\beta(t), z), \quad p = p(t, x, y + \tau\beta(t), z) \\
& u = u(t, x, y + \tau\beta(t), z), \quad v = v(t, x, y + \tau\beta(t), z) - \tau\beta'(t) \\
& w = w(t, x, y + \tau\beta(t), z) \\
(Z_\gamma) \quad & \rho = \rho(t, x, y, z + \tau\gamma(t)), \quad p = p(t, x, y, z + \tau\gamma(t)) \\
& u = u(t, x, y, z + \tau\gamma(t)), \quad v = v(t, x, y, z + \tau\gamma(t)) \\
& w = w(t, x, y, z + \tau\gamma(t)) - \tau\gamma'(t) \\
(P_\varepsilon) \quad & \rho = \rho(t, x, y, z), \quad p = p(t, x, y, z) + \tau\varepsilon(t) \\
& u = u(t, x, y, z), \quad v = v(t, x, y, z) \\
& w = w(t, x, y, z) \\
(PR) \quad & \rho = \rho(t, x, y, z) + \tau, \quad p = p(t, x, y, z) + gz\tau \\
& u = u(t, x, y, z), \quad v = v(t, x, y, z) \\
& w = w(t, x, y, z) \\
(M_1) \quad & \rho = \tau\rho(\tau t, x, y, z), \quad p = \tau p(\tau t, x, y, z) \\
& u = \tau u(\tau t, x, y, z), \quad v = \tau v(\tau t, x, y, z) \\
& w = \tau w(\tau t, x, y, z) \\
(M_2) \quad & \rho = \tau\rho(t, \tau x, \tau y, \tau z), \quad p = p(t, \tau x, \tau y, \tau z) \\
& u = u(t, \tau x, \tau y, \tau z)/\tau, \quad v = v(t, \tau x, \tau y, \tau z)/\tau \\
& w = w(t, \tau x, \tau y, \tau z)/\tau \\
(\Theta_\delta) \quad & \rho = \rho(t, r, \theta - \tau\delta, z), \quad p = p(t, r, \theta - \tau\delta, z) \\
& u = A \cos(\tau)u(t, r, \theta - \tau\delta, z) + B \sin(\tau)v(t, r, \theta - \tau\delta, z) \\
& \quad - \frac{\delta'}{\delta} r \theta \sin \theta \\
& v = -A \sin(\tau)u(t, r, \theta - \tau\delta, z) + B \cos(\tau)v(t, r, \theta - \tau\delta, z) \\
& \quad + \frac{\delta'}{\delta} r \theta \cos \theta \\
& w = w(t, r, \theta - \tau\delta, z)
\end{aligned}$$

are solutions of the system too.

The following quotientequations correspond to the above considered symmetries:

$$\begin{aligned}
 (T) \quad & \frac{\partial p}{\partial x} = \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \\
 & \frac{\partial p}{\partial y} = \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \\
 & \frac{\partial p}{\partial z} = \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + g\rho, \\
 & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\
 & u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = 0,
 \end{aligned}$$

$$\begin{aligned}
 (X_\alpha) \quad & \mu \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0, \\
 & \frac{\partial p}{\partial y} = \mu \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \\
 & \frac{\partial p}{\partial z} = \mu \left( \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + g\rho, \\
 & \frac{\alpha'(t)}{\alpha(t)} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\
 & \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = 0,
 \end{aligned}$$

$$\begin{aligned}
 (Y_\beta) \quad & \frac{\partial p}{\partial x} = \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right), \\
 & \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} \right) = 0, \\
 & \frac{\partial p}{\partial z} = \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) + g\rho, \\
 & \frac{\partial u}{\partial x} + \frac{\beta'(t)}{\beta(t)} + \frac{\partial w}{\partial z} = 0, \\
 & \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} = 0,
 \end{aligned}$$

$$\begin{aligned}
 (Z_\gamma) \quad & \frac{\partial p}{\partial x} = \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \\
 & \frac{\partial p}{\partial y} = \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \\
 & \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + g\rho = 0, \\
 & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\gamma'(t)}{\gamma(t)} = 0,
 \end{aligned}$$

$$\begin{aligned}
(M_1) \quad & \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 0, \\
& \frac{\partial p}{\partial x} = \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \\
& \frac{\partial p}{\partial y} = \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \\
& \frac{\partial p}{\partial z} = \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + g\rho, \\
& \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\
& u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = \rho,
\end{aligned}$$

$$\begin{aligned}
(\Theta_\delta) \quad & v = A(t, r, z) \cos(\theta) + B(t, r, z) \sin(\theta) + \frac{\delta'}{\delta} r \theta \sin \theta, \\
& u = -A(t, r, z) \sin(\theta) + B(t, r, z) \cos(\theta) - \frac{\delta'}{\delta} r \theta \cos \theta, \\
& \frac{\partial p}{\partial r} = \mu \left( \frac{\partial^2 B}{\partial r^2} + \frac{1}{r} \frac{\partial B}{\partial r} - \frac{1}{r^2} B + \frac{\partial^2 B}{\partial z^2} - \frac{\delta'}{\delta} \frac{2}{r} \right), \\
& \frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} - \frac{1}{r^2} \frac{\partial^2 A}{\partial z^2} = 0, \\
& \frac{\partial p}{\partial z} = \mu \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right) + g\rho, \\
& \frac{\partial B}{\partial r} + \frac{1}{r} B + 1 + \frac{\partial w}{\partial z} = 0, \\
& \frac{\partial \rho}{\partial t} + B \frac{\partial \rho}{\partial r} + w \frac{\partial \rho}{\partial z} = 0.
\end{aligned}$$

#### 4. CONSERVATION LAWS

**Theorem 5.** *System (1) has the following conservation laws with 0-order generating functions*

$$\begin{aligned}
\psi_\rho &= (a'(\rho) + b(t))w, \\
\psi_p &= -b'(t)/g, \\
\psi_u &= c(t, x, y, z), \\
\psi_v &= d(t, x, y, z), \\
\psi_w &= a(\rho) + b(t)\rho + e(t, x, y),
\end{aligned}$$

where  $c, d, e$  are solutions of the system

$$\begin{aligned}
\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} &= \frac{\partial e}{\partial x}, \\
\frac{\partial^2 d}{\partial x^2} + \frac{\partial^2 d}{\partial y^2} + \frac{\partial^2 d}{\partial z^2} &= \frac{\partial e}{\partial y},
\end{aligned}$$

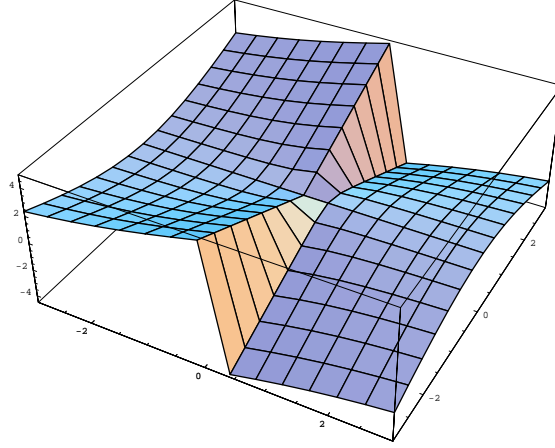


FIGURE 2. The graph of the  $u$  function

$$\frac{\partial c}{\partial x} + \frac{\partial d}{\partial y} = 0.$$

Consider some divergent forms which corresponds to the generating functions in the last theorem:

$$\begin{aligned} (a(\rho))_t + (ua(\rho))_x + (va(\rho))_y + (wa(\rho))_z &= 0, \\ (b(t)\rho)_t + (b(t)u\rho)_x + (b(t)v\rho)_y + (b(t)w\rho)_z + \frac{\mu}{g}\Delta(b'(t)w) \\ - \frac{1}{g}(b'(t)p)_z &= 0. \end{aligned}$$

The main result of the paper is formulated in the following theorem.

**Theorem 6.** *There exist the following higher-order generating functions of the conservation laws*

$$\psi_u = (u_y - v_x)\sigma_y, \quad \psi_v = -(u_y - v_x)\sigma_x,$$

where  $\sigma = (\sigma_x, \sigma_y)$ ,  $\sigma_x + \sigma_y \equiv 1 \pmod{2}$ .

This theorem shown that the system has divergent forms. Moreover, for every  $k$  there exists a divergent form, which contain derivatives of order higher than  $k$ .

## 5. EXAMPLES

Consider the invariant solution of symmetry  $\Theta_\delta$  and let  $\delta = e^t$ ,  $A = 0$ ,  $B = 0$ ,  $\rho = \rho(r)$ ,  $w = w(t, r, z)$ ,  $p = p(t, r)$ . Then

$$\begin{aligned} u &= -r\theta \sin \theta, \\ v &= r\theta \cos \theta, \\ p &= 2\mu \ln \frac{1}{r} + p_0(t), \\ \rho &= 2\rho'_0(r^2), \end{aligned}$$

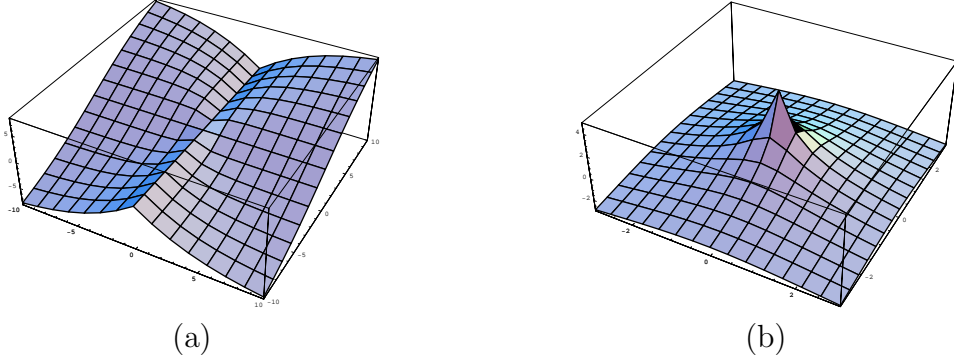


FIGURE 3. The graphs of the  $v$  (a) and  $p$  (b) functions

$$w = -z - \frac{g}{\mu} \int_0^r \frac{\rho(r'^2)}{r'} dr' + w_0(t).$$

The graphs of functions  $u$ ,  $v$  and  $p$  for  $p_0 = 0$ ,  $\mu = 1/2$  are at Fig. 2, and Fig. 3 a,b respectively.

This solution, being subjected to  $PR$ -deformation, transforms to:

$$\begin{aligned} u &= -r\theta \sin \theta = -y \arctan \frac{y}{x}, \\ v &= r\theta \cos \theta = x \arctan \frac{y}{x}, \\ p &= 2\mu \ln \frac{1}{r} + p_0(t) + gz\tau, \\ \rho &= 2\rho'_0(r^2) + \tau, \\ w &= -z - \frac{g}{\mu} \int_0^r \frac{\rho(r'^2)}{r'} dr' + w_0(t). \end{aligned}$$

After  $X_{-t}$ -deformation, we obtain solution with running singularity:

$$\begin{aligned} u &= -y \arctan \frac{y}{x-t} + 1, \\ v &= (x-t) \arctan \frac{y}{x-t}, \\ p &= \mu \ln \frac{1}{(x-t)^2 + y^2} + p_0(t) + gz\tau, \\ \rho &= 2\rho'_0((x-t)^2 + y^2) + \tau, \\ w &= -z - \frac{g}{\mu} \int_0^{\sqrt{(x-t)^2 + y^2}} \frac{\rho(r'^2)}{r'} dr' + w_0(t). \end{aligned}$$

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MOSCOW STATE INSTITUTE ELECTRONICS AND MATHEMATICS, INDEPENDENT UNIVERSITY OF MOSCOW AND INDEPENDENT UNIVERSITY OF MOSCOW, B. VLASIEVSKIJ PER., 11, 121019 MOSCOW, RUSSIA

*E-mail address:* `trushkov@mccme.ru`