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Domains in Infinite Jets: $\mathcal{C}$–Spectral Sequence

A. M. VINOGRADOV AND G. MORENO

Abstract. Domains in infinite jets present the simplest class of diffieties with boundary. In this note some basic elements of geometry of these domains are introduced and an analogue of the $\mathcal{C}$–spectral sequence (see [2, 3]) in this context is studied. This, in particular, allows cohomological interpretation and analysis in the spirit of [1] of initial data, boundary conditions, etc, for general partial differential equations and of transversality conditions in calculus of variations. This kind applications and extensions to arbitrary diffieties will be considered in subsequent publications.

Introduction

The diffiety representing a system of PDEs $\mathcal{E}$ is its infinite prolongation $\mathcal{E}_\infty$, while an initial data problem associated with $\mathcal{E}$ is represented by a subdiffiety of $\mathcal{E}_\infty$ which is, in a sense, the infinite prolongation of the original initial data and whose codimension and co-Dimension are both equal to 1 (see [3]). This is one of numerous situations when the necessity to study a pair of diffieties and, in particular, a diffiety with boundary arises. Such a pair will be denoted $(B, \partial B)$ even when ∂B is not the boundary of B. The simplest situation of this kind is associated with a pair of smooth manifolds $(E, \partial E)$, ∂E being a hypersurface in E, if one puts $B = J^\infty(E, n)$ and

$$\partial B \stackrel{\text{def.}}{=} \{[L]_y^\infty \,|L \subset E, \dim L = n,\ L \text{ intersects } \partial E \text{ transversally at } y\}. \tag{1}$$

In the case when $E \xrightarrow{\pi} M$ is a smooth fiber bundle over an n–dimensional manifold M and $\partial E = \pi^{-1}(\partial M)$, with ∂M being a hypersurface in M, the above construction applied to graphs of sections of π gives $B = J^\infty(\pi)$ and $\partial B = \pi_\infty^{-1}(\partial M)$. This particular case will be referred to as *fibered*. In the most general case, B is an open domain in $J^\infty(E, n)$ and ∂B is *∂–admissible* in B (see [3], 8.4).

In this note the *relative $\mathcal{C}$–spectral sequence of the pair* $(B, \partial B)$, denoted by $(E_r(B, \partial B), d_{r,\mathrm{rel}})$, will be constructed by following the guidelines of [3] (see 12.5).

In the sequel we follow the notation of [1]. Namely, $\mathcal{F}$ and Λ stand for the filtered algebra of smooth functions and for the algebra of differential forms on B, respectively, $\mathcal{C} \subset \Lambda$ for the ideal of Cartan forms, $\overline{\Lambda}$ for the differential algebra of horizontal forms and $\varkappa$ for the Lie algebra of higher symmetries of B. If $B = J^\infty(\pi)$, then the evolutionary derivation, whose generating function is ψ, is denoted by $\ni_\psi$, ℓ_f stands for the universal linearization of $f \in \mathcal{F}$, and so on. Accordingly, symbols $\mathcal{F}(\partial B)$, $\Lambda(\partial B)$, $\mathcal{C}_{\partial B}$, $\overline{\Lambda}(\partial B)$, $\varkappa(\partial B)$, $\ni_\psi^{\partial B}$, and $\ell_f^{\partial B}$ stand for the corresponding objects on the diffiety ∂B. For instance, $\ell_f^{\partial B}$ is a $\mathcal{C}_{\partial B}$–differential operator. The $\mathcal{C}$–spectral sequence for B is denoted by E_r, while the $\mathcal{C}_{\partial B}$–spectral sequence for ∂B by $E_r(\partial B)$.

A suitable for describing the above situation local chart $(x_1, \ldots, x_n)$ on M is such that $\partial M = \{x_n = 0\}$. It is extended to a chart on E by introducing some fiber coordinates $(u^1, \ldots, u^m)$ and then to the standard jet chart $(x_1, \ldots, x_n, u^1, \ldots, u^m, \ldots, u^k_\sigma, \ldots)$ on $J^\infty(\pi)$. ∂B in this chart is given by $\{x_n = 0\}$. Total derivatives D_i, $1 \leq i \leq n$, on $J^\infty(\pi)$ corresponding to this chart are tangent to ∂B if $i < n$. Denote by $\Pi^{(j)}$, $j = 1, \ldots, m$, the projection of $\mathcal{F}(\pi, \pi)$ to its j–th component and put $D^{(j)}_\sigma = D_\sigma \circ \Pi^{(j)}$ with $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{N}_0^n$ being a multi–index. Elements of the standard basis in $\mathbb{N}_0^n$ are denoted by 1_i, $i = 1, \ldots, n$. By fixing volume forms $dx_1 \wedge \cdots \wedge dx_n$ and $dx_1 \wedge \cdots \wedge dx_{n-1}$ on M and ∂M, respectively, we identify $\overline{\Lambda}^n$–valued operators with $\mathcal{F}$–valued and $\overline{\Lambda}^{n-1}(\partial B)$–valued operators with $\mathcal{F}(\partial B)$–valued ones, respectively.

1. The relative $\mathcal{C}$–spectral sequence, general case

Let $\iota_{\partial B} : \partial B \to B$ be the embedding map. Denote by $\mathcal{L} = \Lambda(B, \partial B)$ the ideal $\ker \iota^*_{\partial B}$ of vanishing on ∂B differential forms on B. Then the restriction of the Cartan distribution on B to ∂B is given by the ideal

$$\mathcal{C}_{\partial B} = \frac{\mathcal{C}}{\mathcal{C} \cap \mathcal{L}} \tag{2}$$

of the quotient algebra $\Lambda/\mathcal{L} = \Lambda(\partial B)$.

Proposition 1. *The submodule*

$$E_0^p(B, \partial B) \overset{\text{def.}}{=} \frac{\mathcal{C}^p \cap \mathcal{L} + \mathcal{C}^{p+1}}{\mathcal{C}^{p+1}} \tag{3}$$

of E_0^p is, moreover, a sub–complex of $(E_0^p,\ d_0)$.

Definition 1. *The term $E_0^0(B, \partial B)$ is called the differential algebra of* relative with respect to ∂B horizontal differential forms on B *and is denoted by* $\overline{\Lambda}(B,\, \partial B)$.

Theorem 1. *The quotient $\mathcal{F}$–module $\frac{E_0^p}{E_0^p(B,\partial B)}$ is isomorphic to the $\mathcal{F}(\partial B)$–module $\frac{\mathcal{C}^p_{\partial B}}{\mathcal{C}^{p+1}_{\partial B}}$, i.e., to $E_0^p(\partial B)$.*

This way one gets the short exact sequence of complexes

$$0 \to E_0^p(B, \partial B) \xrightarrow{i} E_0^p \xrightarrow{\alpha} E_0^p(\partial B) \to 0, \tag{4}$$

which leads to the corresponding long exact cohomology sequence

$$\begin{array}{ccc} E_1^p(B,\partial B) & \xrightarrow{\quad H(i) \quad} & E_1^p \\ {}_{\partial}\nwarrow & & \swarrow_{H(\alpha)} \\ & E_1^p(\partial B) & \end{array} \tag{5}$$

where

$$E_1(B,\partial B) \overset{\text{def.}}{=} H(E_0(B,\partial B)). \tag{6}$$

Proposition 2. *If the the one–line theorem (see [1], 4.3.7) holds for both B and ∂B, then it holds also for the relative $\mathcal{C}$–spectral sequence of the pair $(B,\partial B)$.*

In other words, if the term $E_1^{p,q}(B,\partial B)$ is nontrivial, then either $p=0$, or $q=n$. In particular, if $p=0$, then (5) reads

$$\begin{array}{ccc} \overline{H}(B,\partial B) & \xrightarrow{\quad H(i) \quad} & \overline{H} \\ {}_{\overline{\partial}}\nwarrow & & \swarrow_{H(\alpha)} \\ & \overline{H}(\partial B) & \end{array} \tag{7}$$

and is called the *long exact sequence of horizontal de Rham cohomologies* of the pair $(B,\partial B)$. Here $\overline{\partial}$ is the *horizontal coboundary operator*. Under the hypothesis of Proposition 2, sequence (5) for $p>0$ reduces to the short exact sequence

$$0 \longrightarrow E_1^{p,n-1}(\partial B) \xrightarrow{\partial} E_1^{p,n}(B,\partial B) \xrightarrow{H(i)} E_1^{p,n} \longrightarrow 0. \tag{8}$$

2. The relative $\mathcal{C}$–spectral sequence, fibered case

2.1. **Description of ∂B.** Let Δ be a completely integrable distribution on M. Denote by M_x the leaf of the corresponding to Δ foliation that passes through $x \in M$.

Definition 2. *Two (local) sections $s, s' \in \Gamma_{\text{loc}}(\pi)$ are said to be k–th order tangent along Δ at the point x if their restrictions $s_{|M_x}, s'_{|M_x}$ to M_x are k–th order tangent at x in the usual sense.*

Denote by $[s]^k_{\Delta,x}$ the represented by s equivalence class of local sections of π that are k–th order tangent each other along Δ at x and put

$$J^k_\Delta(\pi) \overset{\text{def.}}{=} \{[s]^k_{\Delta,x} \mid s \in \Gamma_{\text{loc}}(\pi),\ x \in M\}, \tag{9}$$

$$\pi^\Delta_{k,l} : J^k_\Delta(\pi) \to J^l_\Delta(\pi), \quad \pi^\Delta_{k,l}\left([s]^k_{\Delta,x}\right) \overset{\text{def.}}{=} [s]^l_{\Delta,x}, \quad k \geq l, \tag{10}$$

$$\pi^\Delta_k : J^k_\Delta(\pi) \to M, \quad \pi^\Delta_k\left([s]^k_{\Delta,x}\right) \overset{\text{def.}}{=} x. \tag{11}$$

The inverse limit $J^\infty_\Delta(\pi) \xrightarrow{\pi^\Delta_\infty} M$ of π^Δ_k, $k \to \infty$, is defined in the usual way.

The following results clarify how $J^\infty_\Delta(\pi)$ is related to $J^\infty(\pi)$ and the infinite jets bundle of the restriction $\pi_{|M_x}$.

Proposition 3. *The maps*

$$J^\infty(\pi_{|M_x}) \to J^\infty_\Delta(\pi), \quad [s]^\infty_x \mapsto [\widetilde{s}]^\infty_{\Delta,x}, \tag{12}$$

with $\widetilde{s}$ being an extension of $s \in \Gamma_{\mathrm{loc}}(\pi_{|M_x})$ to $\Gamma_{\mathrm{loc}}(\pi)$, and

$$J^\infty(\pi) \to J^\infty_\Delta(\pi), \quad [s]^\infty_x \mapsto [s]^\infty_{\Delta,x}, \tag{13}$$

are injective and surjective, respectively.

Remark 1. Observe that (M, Δ) is a diffiety so that it makes sense to consider Δ–differential operators. If π is linear, the sub–functor $\Delta\mathrm{Diff}_k(\Gamma(\pi), \cdot)$ of $\mathrm{Diff}_k(\Gamma(\pi), \cdot)$ is represented by $\mathcal{J}^k_\Delta(\pi) = \Gamma(\pi^k_\Delta)$. The projection of representative objects $\mathcal{J}^k(\pi) \mapsto \mathcal{J}^k_\Delta(\pi)$ corresponds to the natural inclusion of functors $\Delta\mathrm{Diff}_k(\Gamma(\pi), \cdot) \subset \mathrm{Diff}_k(\Gamma(\pi), \cdot)$. This is the meaning of (13). On the other hand, since Δ–differential operators admit restrictions to the leaves of Δ, the natural projection of functors $\Delta\mathrm{Diff}_k(\Gamma(\pi), \cdot) \to \mathrm{Diff}_k(\Gamma(\pi_{|M_x}), \cdot)$ is represented by an injection $\mathcal{J}^k(\pi_{|M_x}) \subset \mathcal{J}^k_\Delta(\pi)$ of the corresponding representative objects. This is the meaning of (12).

Now, let ∇ be a complementary to Δ completely integrable distribution on M. Then $D(\Delta) \oplus D(\nabla) = D(M)$. Assume also that ∂M is a leaf of Δ.

Proposition 4. *Diffieties $J^\infty_\nabla(\pi^\Delta_\infty)$ and $J^\infty_\Delta(\pi^\nabla_\infty)$ are both identified naturally with $J^\infty(\pi)$.*

Put $\pi^\nabla_{\infty,\partial M} \overset{\text{def.}}{=} (\pi^\nabla_\infty)_{|\partial M}$ and, by using Propositions 3 and 4, define the embedding

$$J^\infty(\pi^\nabla_{\infty,\partial M}) \subset J^\infty_\Delta(\pi^\nabla_\infty) = J^\infty(\pi). \tag{14}$$

Fiber bundle $\pi^\nabla_{\infty,\partial M}$ will be referred to as the (infinite) *normal jets bundle* of π with respect to the hypersurface ∂M. The following result sheds light on the structure of ∂B in the fibered case.

Theorem 2. *Let Δ, ∇ and ∂M be as above. Then diffieties ∂B and $J^\infty(\pi^\nabla_{\infty,\partial M})$ are naturally identified.*

A section of π^∇_∞ is described by a vector $\boldsymbol{f} = (\ldots, f^k_i, \ldots)$, $f^k_i \in C^\infty(M)$, and, correspondingly, a section of $\pi^\nabla_{\infty,\partial M}$ is described by a similar vector, with the $f^k_i \in C^\infty(\partial M)$. So, an element θ of $J^\infty(\pi^\nabla_{\infty,\partial M})$ is represented by the vector $(x_1, \ldots, x_{n-1}, \ldots, \frac{\partial^{|\tau|} f^k_i}{\partial x^\tau}(x_1, \ldots, x_{n-1}), \ldots)$, with $k = 1, \ldots, m$, $i \in \mathbb{N}_0$, and $\tau \in \mathbb{N}_0^{n-1}$. Put

$$(u^k_i)_\tau(\theta) \overset{\text{def.}}{=} \frac{\partial^{|\tau|} f^k_i}{\partial x^\tau}(x_1, \ldots, x_{n-1}), \quad \theta = [\boldsymbol{f}]^\infty_{(x_1,\ldots,x_{n-1})}. \tag{15}$$

The embedding $\iota_{\partial B}$ in the considered case is described by

$$\left\{ \begin{array}{rcll} \iota^*_{\partial B}(x_i) & = & x_i, & i = 1, \ldots, n-1, \\ \iota^*_{\partial B}(x_n) & = & 0, & \\ \iota^*_{\partial B}(u^k_\sigma) & = & (u^k_{\sigma_n})_{\sigma - \sigma_n 1_n}, & k = 1, \ldots, m, \sigma \in \mathbb{N}_0^n. \end{array} \right. \tag{16}$$

2.2. **The E_0 term of the relative $\mathcal{C}$–spectral sequence.** Assume that ∂B is given by the equation $\phi = 0$, $\phi = \pi_\infty^*(\varphi)$. Then the following relations are easily checked:

$$\begin{aligned} \Lambda(B,\partial B) &= \phi\Lambda + d\phi\wedge\Lambda, &(17)\\ \overline{\Lambda}(B,\,\partial B) &= \phi\overline{\Lambda} + \overline{d}\phi\wedge\overline{\Lambda}, &(18)\\ E_0^p(B,\partial B) &= \phi E_0^p + \overline{d}\phi\wedge E_0^p. &(19)\end{aligned}$$

Proposition 5. *The following $\mathcal{F}$–modules isomorphism holds:*

$$E_0^p(B,\partial B) \cong \mathcal{C}\mathrm{Diff}\,^{\mathrm{alt.}}_{(p)}(\varkappa,\,\overline{\Lambda}(B,\,\partial B)). \tag{20}$$

Corollary 1. *The quotient $\frac{E_0^p}{E_0^p(B,\partial B)}$ is isomorphic to $\mathcal{C}_{\partial B}\mathrm{Diff}\,^{\mathrm{alt.}}_{(p)}(\varkappa(\partial B),\,\overline{\Lambda}(\partial B))$.*

It is not difficult to see that the module $\Gamma(\pi^\nabla_{\infty,\partial M})$ is locally free. So, an element $\boldsymbol{\psi}$ of $\mathcal{F}(\partial B,\pi^\nabla_{\infty,\partial M})$ can be represented by its generating function $\boldsymbol{\psi} = (\ldots,\psi_i^k,\ldots)$, $\psi_i^k\in\mathcal{F}(\partial B)$, with $k=1,\ldots,m$ and $i\in\mathbb{N}_0$. The higher symmetry of ∂B corresponding to $\boldsymbol{\psi}$ is represented by the evolutionary derivation

$$\ni^{\partial B}_{\boldsymbol{\psi}} \overset{\text{def.}}{=} \sum D_\tau(\psi_i^k)\frac{\partial}{\partial(u_i^k)_\tau},\qquad \tau\in\mathbb{N}_0^{n-1}, i\in\mathbb{N}_0, k=1,\ldots,m. \tag{21}$$

Denote by $\Pi^{(k,i)}$ the projection of the free module $\mathcal{F}(\partial B,\pi^\nabla_{\infty,\partial M})$ onto its (k,i)–th component and put $D_\tau^{(k,i)} = D_\tau\circ\Pi^{(k,i)}$. The above formula now reads

$$\ni^{\partial B}_{\boldsymbol{\psi}} \overset{\text{def.}}{=} \sum D_\tau^{(k,i)}(\boldsymbol{\psi})\frac{\partial}{\partial(u_i^k)_\tau},\qquad \tau\in\mathbb{N}_0^{n-1}, i\in\mathbb{N}_0, k=1,\ldots,m, \tag{22}$$

and allows to introduce the universal linearization operator

$$\ell_g^{\partial B} \overset{\text{def.}}{=} \sum \frac{\partial g}{\partial(u_i^k)_\tau}D_\tau^{(k,i)},\qquad \tau\in\mathbb{N}_0^{n-1}, i\in\mathbb{N}_0, k=1,\ldots,m,\quad g\in\mathcal{F}(\partial B). \tag{23}$$

Proposition 6. *The isomorphism of Corollary 1 between $\frac{E_0^{1,0}}{E_0^{1,0}(B,\partial B)}$ and $\mathcal{C}_{\partial B}\mathrm{Diff}\,(\varkappa(\partial B),\,\mathcal{F}(\partial B))$ sends the equivalence class $[U_1(f)]$ to the $\mathcal{C}_{\partial B}$–differential operator $\ell^{\partial B}_{f_{|\partial B}}$.*

In view of Proposition 5 and Corollary 1 sequence (4) is identified with

$$0\to\mathcal{C}\mathrm{Diff}\,^{\mathrm{alt.}}_{(p)}(\varkappa,\,\overline{\Lambda}(B,\,\partial B))\xrightarrow{i}\mathcal{C}\mathrm{Diff}\,^{\mathrm{alt.}}_{(p)}(\varkappa,\,\overline{\Lambda})\xrightarrow{\boldsymbol{\alpha}}\mathcal{C}_{\partial B}\mathrm{Diff}\,^{\mathrm{alt.}}_{(p)}(\varkappa(\partial B),\,\overline{\Lambda}(\partial B))\to 0. \tag{24}$$

Proposition 7. *The projection $\boldsymbol{\alpha}$ in (24) corresponding to α in (4) is given by*

$$\boldsymbol{\alpha}(\ell_{f^1}\wedge\cdots\wedge\ell_{f^p}\otimes\overline{\omega}) = \ell^{\partial B}_{f^1_{|\partial B}}\wedge\cdots\wedge\ell^{\partial B}_{f^p_{|\partial B}}\otimes\overline{\iota^*_{\partial B}(\omega)} \tag{25}$$

with $f^1,\ldots,f^p\in\mathcal{F}$ and $\overline{\omega}\in\overline{\Lambda}$.

2.3. **The E_1 term of the relative $\mathcal{C}$–spectral sequence.** Proposition 2 holds in the fibered case, so sequence (8) reads

$$0 \to \widehat{\varkappa}(\partial B) \xrightarrow{\partial} \widehat{\varkappa}(B, \partial B) \xrightarrow{H(i)} \widehat{\varkappa} \to 0. \tag{26}$$

The *relative adjoint* to $\varkappa$ module is defined as

$$\widehat{\varkappa}(B, \partial B) \stackrel{\text{def.}}{=} E_1^{1,n}(B, \partial B), \tag{27}$$

and the module product in it is defined by composing $\mathcal{C}$–differential operators with scalars from the right. With respect to this module structure both ∂ and $H(i)$ are $\mathcal{F}$–linear maps.

Since $\overline{\Lambda}^n(B, \partial B) = \overline{\Lambda}^n$,

$$\widehat{\varkappa}(B, \partial B) = \frac{\mathcal{C}\mathrm{Diff}\,(\varkappa, \overline{\Lambda}^n)}{d_0^{1,n-1}\left(\mathcal{C}\mathrm{Diff}\,(\varkappa, \overline{\Lambda}^{n-1}(B, \partial B))\right)}. \tag{28}$$

This puts in evidence that the complex $E_0^1(B, \partial B)$ has common n–cocycles with E_0^1 but less n–coboundaries than it.

Take an element $\vartheta = [\Box]_{\mathrm{im}\, d_{0,\mathrm{rel}}}$ of $\widehat{\varkappa}(B, \partial B)$, and, by using the $\mathcal{C}$–Green formula for $\Box$ (see [1], 4.1.4), decompose ϑ as the sum $\vartheta = h + \vartheta'$, being $h = [\Box^*(1)]_{\mathrm{im}\, d_{0,\mathrm{rel}}}$ and $\vartheta' = [d \circ \Box']_{\mathrm{im}\, d_{0,\mathrm{rel}}}$, with $\Box' \in \mathcal{C}\mathrm{Diff}\,(\varkappa, \overline{\Lambda}^{n-1})$. Even though $\Box'$ is not uniquely determined by $\Box$, so is $d \circ \Box'$. But $H(i)(\vartheta')$ is zero, so there exist an unique $\theta' \in \widehat{\varkappa}(\partial B)$ such that $\overline{\partial}(\theta') = \vartheta'$. This proves the following

Proposition 8. *The map $\vartheta \mapsto \theta'$ splits the sequence (26).*

Thus we can identify ϑ with the pair $(\Box^*(1),\ \theta') \in \widehat{\varkappa} \oplus \widehat{\varkappa}(\partial B)$.

2.4. **The relative Euler operator.**

Definition 3. *The differential $d_{1,\mathrm{rel}}^{0,n} : \overline{H}^n(B, \partial B) \to \widehat{\varkappa}(B, \partial B)$ is called the relative Euler operator and is denoted by $\boldsymbol{E}_{\mathrm{rel}}$.*

By applying the relative Euler operator to a Lagrangian $L = [\varpi] \in \overline{H}^n(B, \partial B)$ one gets the pair $(\ell_{\varpi}^*(1), \theta'_{\varpi})$ accordingly to Proposition 8. On the other hand, the extremality condition for the corresponding variational problem is $\boldsymbol{E}_{\mathrm{rel}}(L) = 0$, i.e., $\ell_{\varpi}^*(1) = 0$ and $\theta'_{\varpi} = 0$. The first of these two conditions is the classical Euler–Lagrange equation corresponding to L, while the second one we shall call the *transversality conditions* for the variational problem for L with free boundary (see [3], 8.5) to be conform with the terminology in the standard variational calculus.

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