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On Ricci flat metrics possessing non-trivial Conformal Killing Algebras

ROSSELLA PISCOPO

In Memory of Prof. G. Rotondaro.

Abstract. We prove that a non-trivial conformal Killing algebra with bidimensional orbits of a $4-$dimensional pseudo-Riemannian manifold is isomorphic to the Rotondaro algebra either of dimension 3, or of 4.

We also give an exact description of Ricci flat 4-metrics that admit a non-trivial Conformal Killing algebra with bidimensional orbits.

1. Introduction.

In this paper our aim is to clarify some natural links between Lie algebras of Killing vector fields and Lie algebras of conformal Killing vector fields of a given pseudo-Riemannian metric g.

Conformal Killing algebras were less studied in literature than Killing ones. This concerns both the theory and applications. Originally our interest for such algebras was motivated by some questions in General Relativity. One of them was: would it be possible to find new reductions and, therefore, new exact solutions of vacuum Einstein equations by extending to conformal Killing algebras the approach of the papers [11], [13], [14] .

Below the Lie algebra of all conformal Killing field (respectively, Killing fields) of a metric g is denoted by $\mathfrak{Conf}(g)$ (respectively $\mathfrak{Kill}(g)$) and the term conformal Killing algebra (respectively Killing algebra) refers to a sub-algebra $\mathcal{G} \subset \mathfrak{Conf}(g)$ (respectively $\mathcal{L} \subset \mathfrak{Kill}(g)$).

If $\mathcal{L}$ is a Killing algebra of a metric g, then, obviously, $\mathcal{L}$ is a conformal Killing algebra for any metric $\widetilde{g} = \lambda g$, λ being a nowhere vanishing function, i.e.,

$$\mathcal{L} \subset \mathfrak{Kill}(g) \Rightarrow \mathcal{L} \subset \mathfrak{Conf}(\lambda g)$$

This remark gives a simple way of constructing conformal Killing algebras and it is natural to call such algebras *trivial*. So, the problem we are interested in is *what are non-trivial conformal Killing algebras.*

In the first part of the article we answer this question for algebras on $4-$dimensional pseudo-Riemannian manifolds with orbits of dimension non greater than two . We were interested in this special case in view of some applications to General Relativity. The obtained result is somehow surprising. Namely, it turns out that in dimension 4

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there exist only two non-equivalent types of non-trivial conformal Killing algebras with $2-$dimensional orbits and they are isomorphic to one of Rotondaro's algebra $\mathfrak{I}_3$ or $\mathfrak{I}_4$.

The second part of the paper is dedicated to finding of all $4-$dimensional Ricci flat metrics admitting a nontrivial conformal Killing algebra with bidimensional orbits. We show that such a metric is essentially unique and corresponds to the conform Killing algebra isomorphic to $\mathfrak{I}_4$. Moreover, it is Klenian and belongs to the class of metrics studied in [11].

Notations and Conventions

• All objects are assumed to be smooth; (M, g) stands for an n–dimensional pseudo-Riemannian manifold;
• By a *metric* (pseudo-metric) g we refers to a non-degenerate symmetric $(0, 2)$ tensor field of arbitrary signature;
• The $C^\infty(M)-$module of vector fields is denoted by $D(M)$. The Lie derivative along a vector field $X \in D(M)$ is denoted by L_X and coordinate vector fields of a local chart $x_1, .., x_n$ by $\partial_{x_1}, .., \partial_{x_n}$. We use "$\lrcorner$" for the insertion of a field $Y \in D(M)$ into a covariant tensor, for instance, $Y \lrcorner g$.
• The distribution spanned by vector fields belonging to the conformal Killing algebra in question is denoted by $\mathcal{D}$.
• Integral submanifolds of $\mathcal{D}$ are called *Killing orbits* or simply *orbits.*
• $\Delta\varphi$ and H^φ stands for the Laplacian and the Hessian of a function φ, respectively.
• We use the adjective "(in)dependent" for $C^\infty(M)-$(in)dependent and "$\mathbb{R}-$(in)dependent" for (in)dependent over reals.

2. Preliminaries.

In this section the necessary general facts are collected.

Lemma 1. *Let X be a vector field on M. If $f \in C^\infty(M)$, then the following formula holds:*

$$L_{fX}(g) = fL_X(g) + X \lrcorner g \cdot df, \tag{2.1}$$

where the dot "$\cdot$" denotes the symmetric product of two differential 1-forms

Proof. Straightforward from the well-known formula:

$$L_Y(g)(A, B) = Y(g(A, B)) - g([Y, A], B) - g(A, [Y, B]),$$

$Y, A, B \in D(M)$. ∎

It follows immediately from (4.1) that:

Lemma 2. *Let $0 \neq X \in D(M)$, $f \in C^\infty(M)$. If X and fX are Killing fields of a metric g, then f is constant.*

If $\dim M \geq 3$ lemma 2 remains valid for Conformal Killing algebras

Lemma 3. *Let $0 \neq X \in D(M), f \in C^{\infty}(M)$. If X and fX are conformal Killing vector fields of a metric g, then f is constant.*

Proof. Let $L_X(g) = 2\sigma g$ and $L_{fX}(g) = 2\lambda g$. Then in view of (2.1)

$$L_{fX}(g) = fL_X(g) + df \cdot X \lrcorner g = 2f\sigma g + df \cdot X \lrcorner g.$$

so,

$$df \cdot X \lrcorner g = 2(\lambda - f\sigma)\, g. \tag{2.2}$$

Since $\operatorname{rank}(g) \geq 3$ and $\operatorname{rank}(df \cdot X \lrcorner g) \leq 2$, (2.2) holds iff $\lambda = f\sigma$ and $df \cdot X \lrcorner g = 0$. But $X \lrcorner g \neq 0$ and hence $df = 0$. ∎

Lemma 4. *Let X, Y and $\alpha X + \beta Y$, $\alpha, \beta \in C^{\infty}(M)$ be Killing fields of a metric g. Then*

(1) *if X and Y are $\mathbb{R}-$independent, then either α and β are functionally independent, or α and β are constant;*
(2) *if X, Y and $\alpha X + \beta Y$ are $\mathbb{R}-$independent, then α and β are functionally independent, and there exists a non vanishing smooth function χ on M such that*

$$X \lrcorner g = \chi d\beta, \qquad Y \lrcorner g = -\chi d\alpha$$

Proof. See [11]. ∎

The following is obvious

Lemma 5. *If $\mathcal{L}$ is a Killing algebra of a metric g, then it is a conformal Killing algebra for any metric $\widetilde{g} = \lambda g$.*

Definition 1. *Let $\mathcal{G}$ be a conformal Killing algebra of a metric $\widetilde{g}$ on a manifold M. $\mathcal{G}$ is called **trivial** if $\mathcal{G} \subset \mathfrak{Kill}(g)$ for a conformal metric g i.e., $\widetilde{g} = \lambda g$, $\lambda \in C^{\infty}(M)$.*

In the second section we shall need a result concerning the existence of (local) solutions of a first order overdetermined system of the form

$$\begin{cases} f_1 = 0 \\ \cdots \\ f_r = 0 \end{cases} \tag{2.3}$$

with $f_i \in C^{\infty}(J^1(M))$ and $c_1, ..., c_r \in \mathbb{R}$.

Recall that the manifold $J^1(M)$ (see [9]) of 1-*jets* of smooth functions on M possesses a canonical contact structure. A *contact field* $X \in D(J^1(M))$ is uniquely determined by the function $f = X \lrcorner U_1$ where $U_1$is a canonical $1-$form on $J^1(M)$ whose local expression is $U_1 = du - \sum_{i=1}^{n} p_i dx_i$, with $x^1, \ldots, x^n, u, p_1, \ldots, p_n$ being standard jet coordinates on $J^1(M)$. The function f is called the *generating function* of X and we put $X = X_f$.

Definition 2. *The generating function of the contact field $[X_f, X_h]$ is denoted by $\{f, h\}$ and called the Jacobi bracket of f and g.* (see [9]).

In other words,

$$[X_f, X_h] = X_{\{f,h\}}, \quad \{f,h\} = [X_f, X_h] \lrcorner U_1$$

Moreover,

$$\{f,h\} = X_f(h) - X_1(f)\, h.$$

Locally $X_1 = \dfrac{\partial}{\partial u}$.

Proposition 1. *If the ideal generated by $f_1, ..., f_r$ is closed with respect to the Jacobi bracket, the system (2.3) admits solution.* (see [9]).

3. Free conformal Killing algebras.

Definition 3. *A Killing algebra $\mathcal{L}$ (resp., conformal Killing algebra) is called* ***free*** *if*

$$X_p = 0 \Leftrightarrow X = 0$$

for a generic point $p \in M$ and $X \in \mathcal{L}$.

In other words, a k–dimensional Conformal Killing algebra is free if its generic orbits are $k-$dimensional.

Proposition 2. *A free conformal Killing algebra is locally trivial.*

Proof. Let $\mathcal{G}$ be a free conformal Killing algebra for a metric $\widetilde{g}$, $(X_i)_{i\in I}$ a basis of $\mathcal{G}$ and $(c_{ij}^k)_{i,j,k\in I}$ the corresponding structure constants, i.e., $[X_i, X_j] = c_{ij}^k X_k$. Then

$$L_{X_i}(\widetilde{g}) = 2\widetilde{\rho}_i g, \quad \rho_i \in C^\infty(M), \; i \in I. \tag{3.1}$$

and

$$\left[L_{X_i}, L_{X_j}\right](\widetilde{g}) = L_{[X_i,X_j]}(\widetilde{g}) = c_{ij}^k L_{X_k}(\widetilde{g}) = 2c_{ij}^k \rho_k \widetilde{g}, \quad \text{for any } i,j \in I.$$

On the other hand, it follows from (3.1) that

$$\begin{aligned} \left[L_{Xi}, L_{X_j}\right](\widetilde{g}) &= 2X_i(\rho_j)\,\widetilde{g} + 2\rho_j L_{X_i}(\widetilde{g}) - 2X_j(\rho_i)\,\widetilde{g} - 2\rho_i L_{X_j}(\widetilde{g}) \\ &= 2\left(X_i(\rho_j) - X_j(\rho_i)\right)\widetilde{g} \end{aligned}$$

and, therefore,

$$X_i(\rho_j) - X_j(\rho_i) = c_{ij}^k \rho_k \tag{3.2}$$

Assume, now, that X_i's are Killing fields for a metric $g = \lambda\widetilde{g}$. This is equivalent to

$$X_i(\lambda) + 2\lambda\rho_i = 0, \quad i \in I. \tag{3.3}$$

If $X_i = a_i^k \dfrac{\partial}{\partial x_k}$, then the system (3.3) reads

$$a_i^k \frac{\partial \lambda}{\partial x_k} + 2\lambda\rho_i = 0, \quad i \in I.$$

Relations (3.3) form a first order over-determined system of a PDE's imposed on λ and one sees that it is a system of type (2.3) with $f_i = a_i^k p_k + 2u\rho_i$ with $(x_1, .., x_n, u, p_1, .., p_k)$ being the standard jet coordinates in J^1M. If φ is a solution of (3.3), then

$$\begin{aligned} [X_i, X_j](\varphi) &= X_i(X_j(\varphi)) - X_j(X_i(\varphi)) = \\ &= -2\rho_j X_i(\varphi) - 2\varphi X_i(\rho_j) + 2\varphi X_j(\rho_i) + 2\rho_i X_j(\varphi) \\ &= 2\varphi(X_j(\rho_i) - X_i(\rho_j)). \quad i, j \in I. \end{aligned} \tag{3.4}$$

Hence the compatibility condition for (3.3) are

$$[X_i, X_j](\lambda) = 2\lambda(X_j(\rho_i) - X_i(\rho_j))$$

In view of (3.2) and commutation relations for fields X_i's the above relations can be rewritten in the form

$$c_{ij}^k X_k(\lambda) + 2c_{ij}^k \rho_k \lambda = 0$$

As it easy to see, these conditions are equivalent to the fact that $\{f_i, f_j\} = 0$ on the submanifold $\{f_i = 0,\ i = 1, .., \dim \mathcal{G}\} \subset J^1M$. In other words, the hypothesis of Proposition 1 for (3.3) and hence this system possesses solution. ∎

Corollary 1. *The dimension of a non-trivial conformal Killing algebra is strictly greater than the dimension of its generic orbits.*

As an immediate consequence of Proposition 2 we see that any 1−dimensional conformal Killing algebra is trivial.

Proposition 3. *Let $\mathcal{G}$ be a conformal Killing algebra with 1−dimensional orbits on a manifold M,* $\dim M \geq 3$. *Then* $\dim \mathcal{G} = 1$ *and hence $\mathcal{G}$ is trivial.*

Proof. Let $0 \neq X \in \mathcal{G} \subset \mathfrak{Conf}(\widetilde{g})$. Then, according to the above remark, $X \in \mathfrak{Kill}(g)$ for a metric $g = \lambda\widetilde{g}$. If $Y \in \mathcal{G}$, then $Y = fX$, $f \in C^\infty(M)$, and $Y \in \mathfrak{Conf}(g)$, i.e. $L_{fX}(g) = 2\sigma g$, $\sigma \in C^\infty(M)$. Now, formula (2.1) gives

$$L_{fX}(g) = X \lrcorner g \cdot df = 2\sigma g.$$

But $X \lrcorner g \cdot df$ is a tensor of rank ≤ 2 if $df \neq 0$, while $\operatorname{rank}(g) = \dim M \geq 3$. Hence, $\sigma = 0$, $df = 0 \Rightarrow f$ is a constant. ∎

Thus non-trivial conformal Killing algebras must have orbits of dimension greater then 2. Below we shall describe all such algebras with bidimensional orbits. According to Proposition 2 their dimension is greater than 2.

4. Non-free conformal Killing algebras with bidimensional orbits

In this section we assume that $\mathcal{G}$ is a $k-$dimensional conformal Killing algebra of a metric $\widetilde{g}$ with 2−dimensional orbits and $\dim M \geq 3$. Let $\mathcal{D}$ be the distribution generated by $\mathcal{G}$. Then, according to Preposition 2, $k > 2$.

Consider the fields $X, Y \in \mathcal{G}$ generating $\mathcal{D}$ and another field $Z \in \mathcal{G}$. Then $Z = \alpha X + \beta Y$ for some $\alpha, \beta \in C^\infty(M)$. By assumption

$$L_Z(\widetilde{g}) = 2\rho\widetilde{g}, \quad L_X(\widetilde{g}) = 2\sigma\widetilde{g}, \quad L_Y(\widetilde{g}) = 2\tau\widetilde{g}. \tag{4.1}$$

for some $\rho, \sigma, \tau \in C^\infty(M)$.

According to formula (2.1) applied $Z = \alpha X + \beta Y$, the first of equalities (4.1) becomes

$$X \lrcorner \widetilde{g} \cdot d\alpha + Y \lrcorner \widetilde{g} \cdot d\beta = \omega\widetilde{g} \tag{4.2}$$

with $\omega = 2(\rho - \alpha\sigma - \beta\tau)$.

Lemma 6. *$i)$ If $\omega \neq 0$, then $3 \leq \dim M \leq 4$ and $\widetilde{g}$ is indefinite if $\dim M = 3$, while it is of signature $(2,2)$ (Klenian metric) if $\dim M = 4$.*
$ii)$ If $\omega = 0$, then the differentials $d\alpha$, $d\beta$ are independent.

Proof. $i)$ Let $\omega \neq 0$. Note that the rank of the symmetric tensor $X \lrcorner (\widetilde{g}) \cdot d\alpha + Y \lrcorner (\widetilde{g}) \cdot d\beta$ is non greater than 4. On the other hand, the rank of $\widetilde{g}$ is equal to $\dim M \geq 3$. Hence, $3 \leq \dim M \leq 4$ and $\mathrm{rank}(\widetilde{g}) = \dim M = 4$ iff the 1−forms $X \lrcorner \widetilde{g}$, $d\alpha$, $Y \lrcorner \widetilde{g}$, $d\beta$ are independent. Otherwise, these forms are dependent and $\dim M = 3$. In this case, a metric of the form (4.2) is indefinite.

$ii)$ Let $\omega = 0$. Then $X \lrcorner (\widetilde{g}) \cdot d\alpha + Y \lrcorner (\widetilde{g}) \cdot d\beta = 0$ and by lemma 4

$$\begin{cases} Y \lrcorner (\widetilde{g}) = -\lambda d\alpha \\ X \lrcorner (\widetilde{g}) = \lambda d\beta \end{cases}, \quad 0 \neq \lambda \in C^\infty(M) \tag{4.3}$$

Since $X \lrcorner (\widetilde{g})$ and $Y \lrcorner (\widetilde{g})$ are independent, $d\alpha$ and $d\beta$ are independent as well. ∎

In what follows we shall limit our discussion to 4−dimensional manifolds, having in mind some physical applications. The 3-dimensional will be discussed separately.

4.1. 4−dimensional pseudo-Riemannian Manifolds.

Let $(M, \widetilde{g})$ be a 4−fold possessing a conformal Killing algebra with bidimensional orbits. Let X, Y and Z be as above. If the function $\omega = \omega_Z$ in (4.2) differs from zero, then 1−forms $X \lrcorner \widetilde{g}$, $Y \lrcorner \widetilde{g}$, $d\alpha$ and $d\beta$ are independent and we get relations

$$\begin{cases} \widetilde{g}(X, X) = 0 \\ \widetilde{g}(X, Y) = 0 \\ X(\beta) = 0 \\ X(\alpha) = 2\omega_Z \end{cases}, \quad \begin{cases} \widetilde{g}(Y, Y) = 0 \\ \widetilde{g}(X, Y) = 0 \\ Y(\alpha) = 0 \\ Y(\beta) = 2\omega_Z \end{cases}, \tag{4.4}$$

by inserting fields X and Y into (4.2) subsequently.

In particular, (4.4) show that $\widetilde{g}$ vanishes when restricted to an orbit of $\mathcal{G}$.

Moreover, in view of (4.4), vector fields X, Y, Z commute as follows

$$\begin{cases} [X, Z] = 2\omega_Z X + \beta [X, Y] \\ [Y, Z] = 2\omega_Z Y - \alpha [X, Y] \end{cases}. \tag{4.5}$$

The following proposition shows that the generators of the tangent distribution $\mathcal{D}$ spans a bidimensional sub-algebra .

Proposition 4. *Let $\widetilde{g}$ be a metric on a manifold M, $\dim M = 4$, and $\mathcal{G} \subset \mathfrak{Conf}(\widetilde{g})$, $\dim \mathcal{G} = k$. Assume that generic orbits of $\mathcal{G}$ are bidimensional. If $X, Y \in \mathcal{G}$ generate the tangent, to the orbits of $\mathcal{G}$, distribution, then X and Y span a sub-algebra.*

Proof. Suppose X, Y do not span a subalgebra, so that $[X, Y] = \alpha X + \beta Y$, with non simultaneously constant smooth functions α, β on M. Specify previous considerations to the field $Z = [X, Y]$ by putting $\omega = \omega_{[X,Y]}$ and analyze, separately, cases $\omega \neq 0$ and $\omega = 0$. In this case (4.5) gives

$$\begin{cases} [X,[X,Y]] = (2\omega + \beta\alpha)X + \beta^2 Y \\ [Y,[X,Y]] = (2\omega - \alpha\beta)Y - \alpha^2 X \end{cases} . \tag{4.6}$$

where ω is defined as in (4.2).

a. Suppose $\omega \neq 0$.

According to (4.6) the fields $[X, [X, Y]]$ and $[Y, [X, Y]]$ are functional combination of X and Y. So, relation (4.2) takes place for each of them with $\omega_1 = \omega_{[X,[X,Y]]}$ and $\omega_2 = \omega_{[Y,[X,Y]]}$, respectively.

• If $\omega_1 \neq 0, \omega_2 \neq 0$. Then, one of relations (4.4) for the field $[X, [X, Y]]$ gives $Y(2\omega + \alpha\beta) = 0$ and similarly, for the field $[Y, [X, Y]]$ one gets give $X(2\omega - \alpha\beta) = 0$. In view of (4.4) for $Z = [X, Y]$ these relation are equivalent to

$$Y(\omega) = -\alpha\omega, \quad X(\omega) = \beta\omega.$$

Now one sees that $X(Y(\omega)) = -2\omega^2 - \alpha\beta\omega$ and $Y(X(\omega)) = 2\omega^2 - \alpha\beta\omega$ and hence $[X, Y](\omega) = -4\omega^2$. On the other hand, $[X, Y](\omega) = \alpha X(\omega) + \beta Y(\omega) = 0$. So, $\omega = 0$ contradicts the assumption.

• If $\omega_1 = 0$, then, according to Lemma 6,

$$\begin{cases} d(2\omega + \beta\alpha) = -\lambda Y \lrcorner (\widetilde{g}) \\ d(\beta^2) = \lambda X \lrcorner (\widetilde{g}) \end{cases} , \quad 0 \neq \lambda \in C^\infty(M).$$

By inserting Y into the second equation, one gets

$$2\beta Y(\beta) = \widetilde{g}(X, Y)$$

Since $\omega \neq 0$, in virtue of (4.4) one has $\widetilde{g}(X, Y) = 0$ and $Y(\beta) = 2\omega$. Hence,

$$2\beta Y(\beta) = 4\omega\beta = 0$$

So, $\beta = 0$ and formula (4.2) becomes $X \lrcorner \widetilde{g} \cdot d\alpha = \omega\widetilde{g}$. But $X \lrcorner g \cdot d\alpha$ is a tensor of rank ≤ 2 if $d\alpha \neq 0$, while rank $(g) = \dim M = 4$. Hence, the contradiction $\omega = 0$, α constant. Similarly, one proves that $\omega_2 \neq 0$.

b. Suppose $\omega = 0$ and pay attention to

$$\begin{cases} [X,[X,Y]] = (X(\alpha) + \beta\alpha)X + (X(\beta) + \beta^2)\,Y \\ [Y,[X,Y]] = (Y(\beta) - \alpha\beta)Y + (Y(\alpha) - \alpha^2)\,X \end{cases} \tag{4.7}$$

Consider relation (4.2) for $[X,[X,Y]]\,,\ [Y,[X,Y]]$, presented functional combinations of fields X and Y according to (4.7). As before, we put $\omega_1 = \omega_{[X,[X,Y]]}$ and $\omega_2 = \omega_{[Y,[X,Y]]}$.

• Let one of ω_i's, say ω_1, be different from zero. Then, in view of (4.4) for $Z = [X,[X,Y]]$, $\widetilde{g}(Y,Y) = \widetilde{g}(X,Y) = \widetilde{g}(X,X) = 0$ and (4.3) implies $X(\alpha) = X(\beta) = Y(\beta) = Y(\alpha) = 0$.

On the other hand, relations (4.4) for $Z = [X,[X,Y]]$ together with the fact that $X(\alpha\beta) = Y(\beta^2) = 0$, shows that $\omega_1 = 0$ in contradiction with the assumption.

Similarly, we get the same result for $\omega_1 = 0,\ \omega_2 \neq 0$.

• Let $\omega_1 = \omega_2 = 0$.

Since $\omega = 0$, (4.3) holds and implies

$$X(\alpha) + Y(\beta) = 0 \tag{4.8}$$

Relations (4.3), for fields $Z = [X,[X,Y]]\,,\ Z = [Y,[X,Y]]$ in view of (4.7), gives

$$\begin{cases} dX(\alpha) + d(\alpha\beta) = -\epsilon Y \lrcorner (\widetilde{g}) \\ dX(\beta) + d\beta^2 = \epsilon X \lrcorner (\widetilde{g}) \\ dY(\alpha) - d\alpha^2 = -\delta Y \lrcorner (\widetilde{g}) \\ dY(\beta) - d\alpha\beta = \delta X \lrcorner (\widetilde{g}) \end{cases}, \quad \begin{matrix} 0 \neq \epsilon \in C^\infty(M) \\ 0 \neq \delta \in C^\infty(M) \end{matrix}$$

Now, by summing up the first and the fourth of the above equations and using (4.8) we get

$$\delta X \lrcorner (\widetilde{g}) - \epsilon Y \lrcorner (\widetilde{g}) = 0.$$

So $\delta = \epsilon = 0$ in contradiction with Lemma 4. ∎

Hence, the conformal Killing algebra $\mathcal{G}$ is such that each its bidimensional subspace is a sub-algebra. Lie algebras possessing this property can be explicitly described.

Theorem 1. *If $\mathcal{G}$ is an $n-$dimensional Lie algebra over a field $\Bbbk$ whose $2-$dimensional subspaces are all subalgebras, then $\mathcal{G}$ is isomorphic either to the abelian algebra $\mathcal{A}_n$, or to*

$$\mathfrak{I}_n = \langle e_1, ..., e_n : [e_i, e_j] = 0,\ i,j \leq n-1,\ [e_n, e_i] = e_i,\ i \leq n-1 \rangle$$

Proof. Obviously, any subspace $V \subset |\mathcal{G}|$ is a subalgebra. The proof will be by induction on $n \geq 3$.

First, consider $\dim \mathcal{G} = n = 3$. If all 2–subspaces are abelian, then $\mathcal{G}$ itself is abelian. Assume that at least one of the subspaces is not abelian. Choose a base in it, say, $\{e,h\}$, such that $[e,h] = h$ and complete it to a base (e,h,f), $f \in |\mathcal{G}|$ in $|\mathcal{G}|$. Since span $\{h,f\}$ is a subalgebra, then $[h,f] = ph + qf,\ p,q \in \Bbbk$, and, similarly, $[e,f] = re + sf,\ r,s \in \Bbbk$. The Jacobi Identity for the triple e,h,f gives

$$qf - (qr)\,e + (sp - r)\,h = 0 \Leftrightarrow q = 0,\ r = ps$$

So,

$$[h, f] = ph, \; [e, f] = s\,(pe + f)\,,$$

Next, the subspace $V = \operatorname{span}(e, h + f)$ is a subalgebra. So, the commutator

$$[e, h + f] = spe + h + sf = spe + (h + f) + (s - 1)f$$

belongs to V as well. This implies that $(s - 1)\; f \in V \Leftrightarrow s = 1$, i.e., $[e, f] = pe + f$. Also $[h, pe + f] = 0$. By putting

$$e_1 = h, \; e_2 = pe + f, \; e_3 = e.$$

one gets the result, i.e. the initial step of the induction is proved.

Assume the result for all $(n-1)$-dimensional algebras and prove it for an n-dimensional algebra $\mathcal{G}$. If any $(n-1)$-dimensional subspace of $\mathcal{G}$ is abelian, then $\mathcal{G}$ is, obviously, abelian. Assume that $V \subset \mathcal{G}$, $\dim V = n - 1$ is not abelian, therefore, isomorphic to $\mathfrak{I}_{n-1}$. Let $V' \subset V$ be the support of the derived (abelian) subalgebra of V and $e \in V$ be such that $ad_e|_{V'} = Id_{V'}$. Consider a subspace $W \subset |\mathcal{G}|$ such that $\dim W = (n-1)\,, e \in W$ and $W \neq V$. Note that $|\mathcal{G}| = \operatorname{span}\{V, W\}$ and that $V \cap W$ is an $(n-2)$ $-$dimensional subalgebra. Since $e \in V \cap W$, codim $(V' \cap W)$ in $V \cap W$ is equal to 1 and, therefore, $\dim(V' \cap W) = n - 3 \geq 1$. Obviously, $ad_e|_{V' \cap W} = Id_{V' \cap W}$; therefore, $V \cap W$ is non-abelian and, by induction, is isomorphic to $\mathfrak{I}_{n-2}$. This implies that W is a non-abelian subalgebra in $\mathcal{G}$ and, as such, is isomorphic to $\mathfrak{I}_{n-1}$. Denote by W' its "abelian" subspace, $\dim W' = n-2$, and note that $V' \cap W \subset W'$. Indeed, W' is the eigenspace of $ad_e|_W$ corresponding to the eigenvalue 1. So, $\dim(V' + W') = n - 1$ and $ad_e|_{V'+W'} = Id_{V'+W'}$. It remains to prove that $V' + W'$ is abelian. Let $x, y \in V' + W'$. Consider the subalgebra $\beta = \operatorname{span}\{x, y, e\}$. Since $\dim \beta \leq 3 < n$, by induction hypothesis, β is isomorphic to $\mathfrak{I}_s, s \leq 3$. As it is easy to see, the eigenspace of $ad_v, v \in \mathfrak{I}_s$, corresponding to the eigenvalue 1 coincides with $\operatorname{span}\{e_1, ..., e_{s-1}\}$, which is the unique abelian subalgebra of $\mathfrak{I}_s$. But elements $x, y \in \beta$ belong to such eigenspace of $ad_e|_\beta$. So $[x, y] = 0$. ∎

Definition 4. *The Lie algebra $\mathfrak{I}_n$ is called the $n-$dimensional* ***Rotondaro algebra****.*

Remark 1. *Observe that any $(n-1)-$dimensional subspace V in $|\mathfrak{I}_n|$ is a subalgebra in $\mathfrak{I}_n$ isomorphic to $\mathfrak{I}_{n-1}$ if $V \neq |\mathfrak{I}'_n|$. In particular, $\mathfrak{I}'_n$ is the only abelian $(n-1)-$dimensional subalgebra of $\mathfrak{I}_n$.*

According to the above results and to Proposition 2 we see that

Lemma 7. *Let $X, Y \in \mathcal{G}$ be generators of the distribution $\mathcal{D}$. Then, there exists a metric g such that $\widetilde{g} = \lambda g$ for which X and Y are Killing fields.*

Consider, now, $\mathbb{R}-$independent vector fields $X, Y \in \mathcal{G}$ generating the distribution $\mathcal{D}$. According to the previous Lemma, there exisists a metric g, defined as above, for which X, Y are Killing vector fields. Since, by hypothesis, the conformal Killing algebra $\mathcal{G}$

is not trivial, there exists, at least, a field $Z = \alpha X + \beta Y$, $\alpha, \beta \in C^\infty(M)$, such that $L_Z(g) = 2\rho g,\ \rho \neq 0$. Now, the formula (4.2) still holds for the metric g

$$X \lrcorner g \cdot d\alpha + Y \lrcorner g \cdot d\beta = 2\rho g. \tag{4.9}$$

In this case $\omega = 2\rho \neq 0$.

Lemma 8. *The non-trivial conformal Killing algebra $\mathcal{G}$ is not abelian.*

Proof. Let $X, Y, Z \in \mathcal{G}$, defined as above. Suppose $\mathcal{G}$ abelian. Then $[X, Y] = 0$ and, according to (4.5),

$$\begin{cases} 0 = [X, Z] = 2\omega X, \\ 0 = [Y, Z] = 2\omega Y, \end{cases} \tag{4.10}$$

So, $\omega = 0$ which is contradictory. ∎

Lemma 9. *Let $\mathcal{G}$ be a conformal Killing algebra of a metric $\widetilde{g}$ with bidimensional orbits . Then $3 \leq \dim \mathcal{G} \leq 4$*

Proof. According to the Theorem 1, the conformal Killing algebra $\mathcal{G}$ is isomorphic to $\mathfrak{I}_n$. Assume $\dim \mathcal{G} > 4$, then there exists at least $5-$dimensional subspace V in $|\mathcal{G}|$. According to Remark 1, V is abelian or isomorphic to $\mathfrak{I}_5$. In both cases, $\mathcal{G}$ would possess an abelian subalgebra of dimension greater than 3.According to Lemma 8, this conform Killing algebra is trivial, i.e., it is a Killing algebra for a metric g. But this is impossible since, according to a result of [12], abelian Killing algebras with bidimensional orbits can not be of dimension > 3. ∎

Summing up the previous results we have proved that a non trivial conformal Killing algebra is isomorphic to the 3 or 4-dimensional Rotondaro algebra. Moreover, in the next section, we shall give some examples of metrics that admit $\mathfrak{I}_3$ and $\mathfrak{I}_4$ as conformal Killing algebra. Hence, the following theorem has been proved:

Theorem 2. *Let $\widetilde{g}$ be a metric on a $4-$dimensional manifold and $\mathcal{G}$ be a non-trivial conformal Killing algebra for $\widetilde{g}$ with bidimensional orbits. Then, $\mathcal{G}$ is isomorphic either to $\mathfrak{I}_3$, to $\mathfrak{I}_4$. Moreover, there exists a metric g such that $\widetilde{g} = \lambda g, \lambda \in C^\infty(M)$, for which the maximal abelian ideal of $\mathcal{G}$ is a Killing algebra.*

5. Ricci flat $4-$metrics.

Let $\widetilde{g}$ and g defined as in Theorem 2. The aim of this section is to find all Ricci-flat metrics $\widetilde{g}$ that allow either $\mathfrak{I}_3$ or $\mathfrak{I}_4$ as non-trivial conformal Killing algebra with bidimensional orbits. According to Theorem 2 the metric g admits either $\mathcal{A}_2$ or $\mathcal{A}_3$ as its Killing algebra with bidimensional orbits and, moreover, g is Klenian and vanishes along these orbits. As it was shown in [11], Prop. 4, any such metric in a suitable local chart (x, y, u, v) is of the form

$$g = 2dxdu + 2adxdv + 2bdydv + rdu^2 + 2pdudv + qdv^2 \tag{5.1}$$

with a, b, r, p, q being arbitrary function in (u, v).
Fields $X = \partial_x, Y = \partial_y$ generate the Killing algebra isomorphic to $\mathcal{A}_2$, for this metric. If, additionally, metrics (5.1) admits a Killing algebra isomorphic to $\mathcal{A}_3$ that extends the algebra, then $\mathcal{A}_3 = \langle \{\partial_x, \partial_y, v\partial_x + \eta(u,v)\partial_y\}\rangle$, with $\eta_u \neq 0$, the metric has the form (5.1) where

$$a = \frac{\eta_v}{\eta_u}, \quad b = -\frac{1}{\eta_u}$$

(see [11], Corollary 3).

It is useful to observe that transformations

$$\sigma : \begin{cases} \widetilde{x} = x - \gamma(u,v) \\ \widetilde{y} = y - \delta(u,v) \\ \widetilde{u} = u \\ \widetilde{v} = v \end{cases}, \qquad \tau : \begin{cases} \widetilde{x} = x \\ \widetilde{y} = y \\ \widetilde{u} = u \\ \widetilde{v} = h(v) \end{cases} \tag{5.2}$$

preserve the form of the metric (5.1).

5.1. **Ricci-flat metrics admitting $\mathfrak{I}_3$ or $\mathfrak{I}_4$ as a Conformal Killing Algebra.** Apply the results of the previous section to $X = \partial_x$, $Y = \partial_y$ and $Z = \alpha X + \beta Y, \alpha, \beta \in C^\infty(M)$. First, from commutation relations $[Z, X] = X$, $[Z, Y] = Y$ one immediately finds that

$$\alpha = -x + \mu(u,v), \quad \beta = -y + \nu(u,v).$$

Now, transformation σ of (5.2), with $\gamma(u,v) = \mu(u,v)$, $\delta(u,v) = \nu(u,v)$ leads to the chart $\{\widetilde{x} = x - \mu(u,v), \widetilde{y} = y - \nu(u,v), \widetilde{u} = u, \widetilde{v} = v\}$ in which $X = \partial_{\widetilde{x}}$, $Y = \partial_{\widetilde{y}}$, $Z = -\widetilde{x}\partial_{\widetilde{x}} - \widetilde{y}\partial_{\widetilde{y}}$, i.e., $\alpha = -\widetilde{x}$, $\beta = -\widetilde{y}$.
Now, by inserting $X = \partial_{\widetilde{x}}$, $Y = \partial_{\widetilde{y}}$, $\alpha = -d\widetilde{x}$, $\beta = -d\widetilde{y}$ in (4.9) one easily finds that $\omega = -1$ and

$$\widetilde{g} = e^{2\varphi(\widetilde{x},\widetilde{y},\widetilde{u},\widetilde{v})}\left(2d\widetilde{x}d\widetilde{u} + 2a(\widetilde{u},\widetilde{v})d\widetilde{x}d\widetilde{v} + 2b(\widetilde{u},\widetilde{v})d\widetilde{y}d\widetilde{v}\right) \tag{5.3}$$

Thus our problem is reduced to description of Ricci-flat metrics of the form (5.3). This is equivalent to solve the system of differential equations $\widetilde{Ric} = 0$, $\widetilde{Ric}$ being the Ricci tensor of $\widetilde{g}$, with respect to the unknown functions φ, a, b. It is easy to verify that the Ricci tensor of the metric

$$g = 2d\widetilde{x}d\widetilde{u} + 2a(\widetilde{u},\widetilde{v})d\widetilde{x}d\widetilde{v} + 2b(\widetilde{u},\widetilde{v})d\widetilde{y}d\widetilde{v}$$

has only three eventually nonvanishing components, namely, $Ric_{33}, Ric_{34}, Ric_{44}$. By this reason the following relation (see [1])

$$\widetilde{Ric} = Ric - 2H^\varphi + 2d\varphi \otimes d\varphi + \left(\Delta\varphi - 2\left\|grad\varphi\right\|^2\right) g \tag{5.4}$$

where$H^\varphi, \Delta\varphi$ are the Hessian and the Laplacian of the function φ respectively, is useful for further computations.

Among the equations $\widetilde{Ric}_{ij} = 0$ the simplest are those with $(i,j) = (1,1), (1,2), (1,4), (2,2)$ and $(2,3)$. It is not difficult to deduce from them that φ depends only on $\widetilde{u}$ and $\widetilde{v}$.

Now a direct computation, based on (5.4), shows that the only non-zero identically coefficients of $\widetilde{Ric}$ are $\widetilde{Ric}_{33}$, $\widetilde{Ric}_{34}$, $\widetilde{Ric}_{44}$. The corresponding equations $\widetilde{Ric}_{ij} = 0$ can be explicitly solved by a routine procedure and one gets:

$$a = e^{-2\varphi}\left(G(\widetilde{v}) - \frac{F(\widetilde{v})\int e^{2\varphi}d\widetilde{u}}{\left(\int \frac{F(\widetilde{v})}{2}d\widetilde{v} - C\right)} + \int 2e^{2\varphi}\frac{\partial\varphi}{\partial\widetilde{v}}d\widetilde{u}\right)$$
$$b = e^{-2\varphi}F(\widetilde{v})$$

Observe, now, that transformation τ of (5.2), with $\overline{v} = h(\widetilde{v}) = \int F(\widetilde{v})\, d\widetilde{v}$, brings $\widetilde{g}$ to the form

$$\widetilde{g} = 2\Phi_{\overline{u}}d\overline{x}d\overline{u} + 2\left(\frac{G(\overline{v})}{F(\overline{v})} - \frac{\Phi}{\left(\frac{\overline{v}}{2} - C\right)} + \Phi_{\overline{v}}\right)d\overline{x}d\overline{v} + 2d\overline{y}d\overline{v};$$

with $\Phi(\overline{u}, \overline{v}) = \int e^{2\varphi}d\overline{u}$. A further simplification one obtains by passing to the adapted chart $\{\widehat{x} = 2\overline{x},\ \widehat{y} = 2\overline{y},\ \widehat{u} = \overline{u},\ \widehat{v} = \overline{v} - 2C,\ \}$ and putting $f(\widehat{v}) = \dfrac{G(\overline{v})}{F(\overline{v})}$, $\Psi = \Phi + \dfrac{\int f(\widehat{v})\,\widehat{v}d\widehat{v} + c_1}{\widehat{v}}$,:

$$\widetilde{g} = \Psi_{\widehat{u}}d\widehat{x}d\widehat{u} + \left(\Psi_{\widehat{v}} - \frac{2\Psi}{\widehat{v}}\right)d\widehat{x}d\widehat{v} + d\widehat{y}d\widehat{v} \tag{5.5}$$

and after that to $\left\{\overline{\overline{x}} = \widehat{x},\ \overline{\overline{y}} = \widehat{y},\ \overline{\overline{u}} = \Psi(\widehat{u}, \widehat{v}),\ \overline{\overline{v}} = \widehat{v}\right\}$, :

$$\widetilde{g} = d\overline{\overline{x}}d\overline{\overline{u}} - \frac{2\overline{\overline{u}}}{\overline{\overline{v}}}d\overline{\overline{x}}d\overline{\overline{v}} + d\overline{\overline{y}}d\overline{\overline{v}} \tag{5.6}$$

This proves the following

Proposition 5. *A Ricci-flat metric admitting $\mathfrak{I}_3$ as its Conform Killing algebra is locally equivalent to metric (5.6).*

An interesting feature of metric (5.6) is that it allows a lager Conform Killing algebra with bidimensional orbits than the originally assumed 3-dimensional one. By theorem 2 this algebra is, inevitably, isomorphic to $\mathfrak{I}_4$ and is spanned, as a direct computation shows, by fields,

$$\partial_{\overline{\overline{x}}},\quad \partial_{\overline{\overline{y}}},\quad \overline{\overline{x}}\partial_{\overline{\overline{x}}} + \overline{\overline{y}}\partial_{\overline{\overline{y}}},\quad \frac{1}{\overline{\overline{v}}}\partial_{\overline{\overline{x}}} + \frac{\overline{\overline{u}}}{\overline{\overline{v}}^2}\partial_{\overline{\overline{y}}}\ .$$

Hence, we have proved that

Proposition 6. *If a Ricci-flat metric admits a nontrivial conform Killing algebra with bidimensional orbits, then such an algebra extends to a conform Killing algebra with bidimensional orbits isomorphic to* $\mathfrak{I}_4$.

For completeness it is worth mentioning that the algebra $\mathfrak{Conf}\,(g)$, for metric (5.6) is spanned by fields

	Symmetries of the metric $\widetilde{g}$	*Conformal Term*
1.	$\partial_{\bar{\bar{x}}}$	$\omega = 0$
2.	$\partial_{\bar{\bar{y}}}$	$\omega = 0$
3.	$\bar{\bar{v}}^2 \partial_{\bar{\bar{u}}}$	$\omega = 0$
4.	$\bar{\bar{x}}\partial_{\bar{\bar{y}}} + \bar{\bar{v}}\partial_{\bar{\bar{u}}}$	$\omega = 0$
5.	$\frac{1}{\bar{\bar{v}}}\partial_{\bar{\bar{x}}} + \frac{\bar{\bar{u}}}{\bar{\bar{v}}^2}\partial_{\bar{\bar{y}}}$	$\omega = 0$
6.	$-\bar{\bar{y}}\partial_{\bar{\bar{y}}} + \bar{\bar{v}}\partial_{\bar{\bar{v}}}$	$\omega = 0$
7.	$\bar{\bar{x}}\partial_{\bar{\bar{x}}} - \bar{\bar{y}}\partial_{\bar{\bar{u}}},$	$\omega = 0$
8.	$\frac{\bar{\bar{x}}}{\bar{\bar{v}}}\partial_{\bar{\bar{x}}} + \frac{\overline{\overline{ux}}}{\bar{\bar{v}}^2}\partial_{\bar{\bar{y}}} - \frac{\bar{\bar{u}}}{\bar{\bar{v}}}\partial_{\bar{\bar{u}}} - \partial_{\bar{\bar{v}}}$	$\omega = 0$
9.	$\frac{\bar{\bar{y}}}{\bar{\bar{v}}}\partial_{\bar{\bar{x}}} + \frac{\overline{\overline{uy}}}{\bar{\bar{v}}^2}\partial_{\bar{\bar{y}}} - \frac{\bar{\bar{u}}^2}{\bar{\bar{v}}^2}\partial_{\bar{\bar{u}}} - \frac{\bar{\bar{u}}}{\bar{\bar{v}}}\partial_{\bar{\bar{v}}},$	$\omega = 0$
10.	$\frac{1}{2}\bar{\bar{x}}^2\partial_{\bar{\bar{x}}} + \frac{1}{2}\overline{\overline{xy}}\partial_{\bar{\bar{y}}} + \left(\frac{\overline{\overline{yv}}^3 - 2\overline{\overline{uxv}}^2}{2\bar{\bar{v}}^2}\right)\partial_{\bar{\bar{u}}} - \frac{\overline{\overline{vx}}}{2}\partial_{\bar{\bar{v}}}$	$\omega = 0$
11.	$\bar{\bar{y}}\partial_{\bar{\bar{x}}} - \frac{2\bar{\bar{u}}^2}{\bar{\bar{v}}}\partial_{\bar{\bar{u}}} - \bar{\bar{u}}\partial_{\bar{\bar{v}}}$	$\omega = -\frac{2\bar{\bar{u}}}{\bar{\bar{v}}}$
12.	$\overline{\overline{xy}}\partial_{\bar{\bar{x}}} + \bar{\bar{y}}^2\partial_{\bar{\bar{y}}} + \left(\overline{\overline{uy}} - 2\frac{\overline{\overline{xu}}^2}{\bar{\bar{v}}}\right)\partial_{\bar{\bar{u}}} - \overline{\overline{ux}}\partial_{\bar{\bar{v}}}$	$\omega = \frac{2\overline{\overline{yv}} - 2\overline{\overline{xu}}}{\bar{\bar{v}}}$
13.	$2\overline{\overline{uv}}\partial_{\bar{\bar{u}}} + \bar{\bar{v}}^2\partial_{\bar{\bar{v}}}$	$\omega = 2\bar{\bar{v}}$
14.	$\left(-\overline{\overline{yv}}^2 + 2\overline{\overline{xuv}}\right)\partial_{\bar{\bar{u}}} + \overline{\overline{xv}}^2\partial_{\bar{\bar{v}}}$	$\omega = 2\overline{\overline{xv}}$
15.	$\bar{\bar{y}}\partial_{\bar{\bar{y}}} + \bar{\bar{u}}\partial_{\bar{\bar{u}}}$	$\omega = 1$

It is clear that the first 10 fields span the Killing algebra of the metric $\widetilde{g}$,

$\mathfrak{Kill}\,(\widetilde{g}) \subset \mathfrak{Conf}\,(\widetilde{g})$.

Obviously, the algebra $\mathfrak{Kill}\,(\widetilde{g})$ (and therefore, $\mathfrak{Conf}\,(\widetilde{g})$) acts transitively on M and metric (5.6) can be realized globally as an invariant metric on the homogeneous space of the corresponding Lie group.

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