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# On multiple generalizations of Lie algebras and Poisson manifolds

Alexandre VINOGRADOV, Michael VINOGRADOV

*In memory of M. Ya. Firer*

Abstract. The notion of $(n, k, r)$-Lie algebra ($n > k \geqslant r \geqslant 0$), an $n$-ary generalization of that of Lie algebra, is introduced and studied. The standard Lie algebras turn out to be $(2, 1, 0)$-Lie algebras. Two types of $n$-ary Lie structures studied in recent few years in the context of the Nambu and "non-Nambu" generalizations of dynamics correspond to $(n, n-1, 0)$- and $(n, 1, 0)$-Lie algebras, respectively.

## Introduction

In last few years interest to $n$-ary generalizations of the concept of Lie algebra and, in particular, of that of Poisson manifold has been growing. This is due to various speculations concerning dynamics and to mathematical curiosity as well. We skip the discussion of motivations, which can be found in the cited literature.

Up to now two different generalizations of the notion of Lie algebra were proposed (see below). They correspond to different readings of the standard Jacobi identity and have quite different properties. The natural question "which of them is better for dynamical or algebraic purposes?" actual a few years ago, is now overcome: they both have their own area of applications. In this paper we propose unifying point of view. Namely, a three-parameter family of multiple Lie algebras is defined in such a way that the mentioned structures appear as its particular subcases.

Below by an $n$-ary Lie algebra structure we understand an $n$-ary multi-linear and skew symmetric operation satisfying a kind of Jacobi identity. Since the meaning of terms "multi-linear" and "skew symmetric" is obvious, the crucial point is what should an $n$-ary generalization of the standard Jacobi identity be.

Historically, the first work (as far as we know) dedicated to this topics was V. T. Filippov's paper [6] of 1985. In it, V. T. Filippov considers $n$-ary operations $[a_1, \ldots, a_n]$ subject to the following Jacobi-like identity.

$$[a_1, \ldots, a_{n-1}, [b_1, \ldots, b_n]] = \sum [b_1, \ldots, b_{i-1}[a_1, \ldots, a_{n-1}, b_i], \ldots, b_n]. \tag{J1}$$

with $a_i, b_j$ belonging to the base vector space $A$. This identity tells us that the "adjoint representation" $a \mapsto [a_1, \ldots, a_{n-1}, a]$ is a derivation of the structure in question for any $a_1, \ldots, a_{n-1} \in A$.

In this paper V. T. Filippov gave the first basic example and developed structural notions such as simplicity and nilpotency in that context. Also he classified $n$-Lie algebras of dimension $n+1$. This classification is parallel to the Bianchi classification of 3-dimensional Lie algebras. Following V. T. Filippov, Sh. M. Kasymov in [10, 11, 12] introduced the notions of $k$-solvability, $k$-nilpotency ($0 < k < n$), and of Cartan subalgebra for Filippov $n$-Lie algebras. He proved also an $n$-ary analog of the Engel theorem.

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A (J1)-Lie algebra structure, realized as a multi-derivation of the smooth function algebra on a smooth manifold, is an $n$-Poisson structure (of type (J1)). The functional determinant

$$\{f_1, \ldots, f_n\} = \det \left\| \frac{\partial f_i}{\partial x_j} \right\|. \tag{0.1}$$

is the simplest example of such a structure in $A = C^\infty(\mathbb{R}^n)$. This was already noticed by V. T. Filippov in his paper cited above. However, neither V. T. Filippov himself nor his followers paid any special attention to this particular case. This was done by "physicists".

Namely, in 1973 Y. Nambu [19] proposed a generalization of Hamiltonian dynamics in which the standard (binary) Poisson bracket was replaced by the ternary one given by (0.1) This generalization is now known as the Nambu dynamics. However, Y. Nambu himself as well as his followers did not mention that the $n$-bracket (0.1) is subject to the $n$-Jacobi identity (J1). This was noticed much later, in 1992, by D. Sahoo and M. C. Valsakumar [22]. A number of persons, apart from V. T. Filippov claim this discovery by referring to their private communications. This discovery was the starting point for L. Takhtajan [23], who systematically developed the fundamentals of $n$-Poisson manifolds (of type (J1)) and called them Nambu – Poisson manifolds. Filippov's work [6], it seems, was unnoticed by the mathematical physics audience.

By assuming an $n$-ary operation of type (J1) on the smooth function algebra $C^\infty(M)$ to be local, one obtains structures more general than $n$-Poisson one. They were called $n$-Jacobi structures and studied by G. Marmo, G. Vilasi, and A. Vinogradov in [17].

Another natural $n$-ary generalization of the standard Jacobi identity has the form

$$\sum_{\substack{i_1<\cdots<i_n \\ j_1<\cdots<j_{n-1}}} (-1)^{(i_1,\ldots,i_n,j_1,\ldots,j_{n-1})}[[a_{i_1},\ldots,a_{i_n}],a_{j_1},\ldots,a_{j_{n-1}}] = 0, \tag{J2}$$

where the sum is taken over all ordered multi-indexes $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_{n-1})$ such that

$$\{i_1, \ldots, i_n\} \bigcup \{j_1, \ldots, j_{n-1}\} = \{1, 2, \ldots, 2n-1\}.$$

In their notes of 1990 – 92 which were published much later [18], P. Michor and A. Vinogradov introduced the alternative notion of $n$-ary Lie algebras whose operations are subject to (J2). The authors were aware of Filippov's work [6] and observed that (J1)-$n$-Lie algebras are also (J2)-$n$-Lie algebras for even $n$. A standard set of accompanying notions: modules, Hochschild cohomology, etc, was also considered there.

Independently this kind of multiple Lie algebra, together with the associated Koszul-type homologies, was introduced by P. Hanlon and M. L. Wachs [9]. This concept was studied by V. Gnedbaye [8] in the context of operads. Note also that (J2)-structures are intimately related to the Schlesinger – Stasheff homotopy algebras [24] and SH-algebras [15].

The corresponding type of Poisson manifolds of even multiplicity was first considered by J. A. de Azcarraga, A. M. Perelomov, and J. C. Pérez-Bueno [3]. These authors treat $n$-Poisson structures as given by $n$-vector fields $V$ such that $[\![V, V]\!] = 0$, where $[\![\cdot, \cdot]\!]$ is the Schouten – Nijenhuis bracket. See papers [2, 4, 20] for further developments.

$n$-Lie and $n$-Poisson structures of types (J1) and (J2) possess rather different properties. One of them is worth mentioning here. Namely, it was conjectured by L. Takhtajan [23] that the structure tensor of a Nambu – Poisson manifold of multiplicity $n$, i.e. of a (J1)-$n$-Poisson one, is of rank $n$ (= locally decomposable) if

$n > 2$. This conjecture was proved first by D. Alekseevsky and P. Guha [1], and a little bit later by P. Gautheron [7] and A Panov [21]. A similar result for $n$-Jacobi manifolds of type (J1) is proved by G. Marmo, G. Vilasi, and A. Vinogradov in [17]. On the contrary, the rank and the multiplicity of an (J2)-Poisson manifold are absolutely independent.

Let us also note a singular behavior of $n$-Lie and $n$-Poisson structures as defined inside the (J2)-type. For instance, since $[\![V,V]\!]$ is identically equal to zero for any $n$-vector field with odd $n$, the approach of [3] is no longer valid to define (J2)-$n$-Poisson manifolds for odd $n$.

The main goal of these notes is to find a general scheme for which (J1) and (J2) types would be particular cases. Our proposal is formulated in Section 3, where $(n,k,r)$-Lie algebras and the corresponding Poisson manifolds are defined and some their basic properties are discussed. It turns out that (J1)-type Lie algebras are $(n,n-1,0)$-Lie algebras, while (J2)-type is formed by $(n,1,0)$-Lie algebras. Section 1 contains the necessary algebraic preliminaries. Two simple facts that make passage from $n$-Lie algebras to $n$-Poisson manifolds automatic are reported in Section 2.

At present the importance of the proposed construction for dynamics it is not clear. But from the mathematical point of view, it is an advantage to possess an algebra of multiple Lie algebras rather than isolated Lie algebra types.

Below, for simplicity we consider the non-graded case only and do not touch homological aspects. This will be covered in a forthcoming systematic account.

## 1. Action and Richardson – Nijenhuis bracket

1.1. **Notation.** Set $\mathrm{I}^{(n)} \overset{\text{def}}{=} (1,\dots,n)$. We will use the letters $I$, $J$ to denote ordered multi-indexes $I=(i_1,\dots,i_k)$, $1\leqslant i_1<i_2<\ldots<i_k\leqslant n$, $J=(j_1,\dots,j_l)$, $1\leqslant j_1<j_2<\ldots<j_l\leqslant n$. Let $|I|=k$, $|J|=l$. If $i_r\neq j_s$ for all $r\leqslant k,\ s\leqslant l$, then a non-ordered multi-index $(I,J)\overset{\text{def}}{=}(i_1,\dots,i_k,j_1,\dots,j_l)$ is defined. We will denote by $(-1)^{(I,J)}$ the parity of the corresponding permutation and by $I+J$ the ordering of multi-index $(I,J)$. The multi-index $(1,2,\dots,i-1,i+1,\dots,n)$ will be denoted by $\mathrm{I}^{(n)}-i$.

1.2. **Main definitions.** Let $A$ be a vector space over a field $K$; denote by $\mathrm{Alt}^n_K A$ the set of all skew-symmetric $n$-linear maps from $A$ to $A$. Let $\mathrm{Alt}^*_K A=\bigoplus_{n=1}^{\infty}\mathrm{Alt}^n_K A$.

**Definition 1.1.** We consider a map $\mathcal{L}\in\mathrm{Alt}^l_K A$. Let the *action* $\mathcal{L}[\cdot]$ be the map

$$\mathcal{L}[\cdot]:\mathrm{Alt}^n_K A\longrightarrow\mathrm{Alt}^{n+l-1}_K A,$$

such that

$$\mathcal{L}[\mathcal{N}](a_{\mathrm{I}^{(l+n-1)}})=\sum_{\substack{I+J=\mathrm{I}^{(l+n-1)}\\ |I|=n,\ |J|=l-1}}(-1)^{(I,J)}\mathcal{L}(\mathcal{N}(a_I),a_J),\tag{1.1}$$

where $a_{\mathrm{I}^{(l+n-1)}}=(a_1,\dots,a_{l+n-1})$, $a_I=(a_{i_1},\dots,a_{i_n})$, $a_J=(a_{j_1},\dots,a_{j_{l-1}})$, and $a_i\in A$.

**Definition 1.2.** The *Richardson – Nijenhuis* bracket $[\![\mathcal{L},\mathcal{N}]\!]^{\mathrm{RN}}\in\mathrm{Alt}^{l+n-1}_K A$ of maps $\mathcal{L}\in\mathrm{Alt}^l_K A$, $\mathcal{N}\in\mathrm{Alt}^n_K A$ is defined by

$$[\![\mathcal{L},\mathcal{N}]\!]^{\mathrm{RN}}\overset{\text{def}}{=}(-1)^{(l-1)(n-1)}\mathcal{L}[\mathcal{N}]-\mathcal{N}[\mathcal{L}]\tag{1.2}$$

Let $\mathcal{L}\in\mathrm{Alt}^l_K A$ and $a_1,\dots,a_k\in A$. Recall that the map $\mathcal{L}_{a_1,\dots,a_k}\in\mathrm{Alt}^{l-k}_K A$ is given by

$$\mathcal{L}_{a_1,\dots,a_k}(b_1,\dots,b_{l-k})\overset{\text{def}}{=}\mathcal{L}(a_1,\dots,a_k,b_1,\dots,b_{l-k}).$$

**Proposition 1.1.** *Let* $\mathcal{L} \in \mathrm{Alt}^l_K A$, $\mathcal{N} \in \mathrm{Alt}^n_K A$ *and* $a \in A$. *Then*

$$\mathcal{L}[\mathcal{N}]_a = \mathcal{L}[\mathcal{N}_a] + (-1)^{n-1}\mathcal{L}_a[\mathcal{N}] \tag{1.3}$$

*Proof.* The proof is by direct calculation. □

**Corollary 1.1.** *Under the conditions of the previous Proposition, we have*

$$[\![\mathcal{L},\mathcal{N}]\!]^{\mathrm{RN}}_a = (-1)^{l-1}[\![\mathcal{L},\mathcal{N}_a]\!]^{\mathrm{RN}} + [\![\mathcal{L}_a,\mathcal{N}]\!]^{\mathrm{RN}} \tag{1.4}$$

*Proof.* Indeed,

$$[\![\mathcal{L},\mathcal{N}]\!]^{\mathrm{RN}}_a = (-1)^{(l-1)(n-1)}\mathcal{L}[\mathcal{N}]_a - \mathcal{N}[\mathcal{L}]_a =$$
$$(-1)^{(l-1)(n-1)}\mathcal{L}[\mathcal{N}_a] + (-1)^{(l-1)(n-1)+(n-1)}\mathcal{L}_a[\mathcal{N}] - \mathcal{N}[\mathcal{L}_a] - (-1)^{(l-1)}\mathcal{N}_a[\mathcal{L}] =$$
$$(-1)^{l-1}((-1)^{(l-1)(n-2)}\mathcal{L}[\mathcal{N}_a] - \mathcal{N}_a[\mathcal{L}]) + (-1)^{(l-1)(n-1)+(n-1)}\mathcal{L}_a[\mathcal{N}] - \mathcal{N}[\mathcal{L}_a] =$$
$$(-1)^{l-1}[\![\mathcal{L},\mathcal{N}_a]\!]^{\mathrm{RN}} + [\![\mathcal{L}_a,\mathcal{N}]\!]^{\mathrm{RN}}.$$

□

1.3. **Exterior multiplication.** Now denote by $\mathrm{Alt}^n_K(A,K)$ the set of all $n$-covectors, i.e., skew-symmetric $n$-linear maps from $A$ to $K$. To each $\varphi \in \mathrm{Alt}^{|\varphi|}_K(A,K)$, $\psi \in \mathrm{Alt}^{|\psi|}_K(A,K)$, $\mathcal{L} \in, \mathrm{Alt}^l_K A$ we assign

$$\varphi \wedge \psi \in \mathrm{Alt}^{|\varphi|+|\psi|}_K(A,K),\ \ \varphi[\mathcal{L}] \in \mathrm{Alt}^{|\varphi|+l-1}_K(A,K),\ \ \varphi \wedge \mathcal{L} \in \mathrm{Alt}^{|\varphi|+l}_K A$$

by the rules

$$(\varphi \wedge \psi)(a_{\mathrm{I}^{(|\varphi|+|\psi|)}}) \stackrel{\mathrm{def}}{=} \sum_{\substack{I+J=\mathrm{I}^{(|\varphi|+|\psi|)}\\ |I|=|\varphi|,\ |J|=|\psi|}} (-1)^{(I,J)}\varphi(a_I)\psi(a_J) \tag{1.5}$$

$$(\varphi \wedge \mathcal{L})(a_{\mathrm{I}^{(|\varphi|+l)}}) \stackrel{\mathrm{def}}{=} \sum_{\substack{I+J=\mathrm{I}^{(|\varphi|+l)}\\ |I|=|\varphi|,\ |J|=l}} (-1)^{(I,J)}\varphi(a_I)\mathcal{L}(a_J) \tag{1.6}$$

$$\varphi[\mathcal{L}](a_{\mathrm{I}^{(|\varphi|+l-1)}}) \stackrel{\mathrm{def}}{=} \sum_{\substack{I+J=\mathrm{I}^{(|\varphi|+l-1)}\\ |I|=|\varphi|,\ |J|=l-1}} (-1)^{(I,J)}\varphi(\mathcal{L}(a_I), a_J). \tag{1.7}$$

Let the vector space $A$ be a $K$-algebra; by analogy with (1.5), we introduce an exterior multiplication in $\mathrm{Alt}^*_K A$. Namely, we define the exterior product $\mathcal{L} \wedge \mathcal{N}$ of $\mathcal{L} \in \mathrm{Alt}^l_K A$ and $\mathcal{N} \in \mathrm{Alt}^n_K A$ by the rule

$$(\mathcal{L} \wedge \mathcal{N})(a_{\mathrm{I}^{(l+n)}}) \stackrel{\mathrm{def}}{=} \sum_{\substack{I+J=\mathrm{I}^{(l+n)}\\ |I|=l,\ |J|=n}} (-1)^{(I,J)}\mathcal{L}(a_I)\mathcal{N}(a_J) \tag{1.8}$$

Obviously, $\mathcal{L} \wedge \mathcal{N} = (-1)^{ln}\mathcal{N} \wedge \mathcal{L}$.

**Lemma 1.1.** $(\mathcal{L} \wedge \mathcal{N})_a = \mathcal{L}_a \wedge \mathcal{N} + (-1)^l \mathcal{L} \wedge \mathcal{N}_a$

*Proof.* It follows immediately from the definitions. □

Denote by $\mathrm{Md}^l_K(A) \subset \mathrm{Alt}^l_K A$ the set of all skew-symmetric multi-derivations, i.e., the set of all elements $\mathcal{L} \in \mathrm{Alt}^L_K A$ satisfying the Leibniz rule:

$$\mathcal{L}(ab, a_2, \ldots, a_l) = a\mathcal{L}(b, a_2, \ldots, a_l) + b\mathcal{L}(a, a_2, \ldots, a_l).$$

Set $\mathrm{Md}^*_K(A) = \bigoplus_l \mathrm{Md}^L_K(A)$.

The following two lemmas are needed for the sequel and proved by direct (but tedious) calculations.

**Lemma 1.2.** *If* $\mathcal{L} \in \mathrm{Md}^l_K(A)$, $\mathcal{P} \in \mathrm{Alt}^p_K A$, $\mathcal{Q} \in \mathrm{Alt}^q_K A$, *then*

$$\mathcal{L}[\mathcal{P} \wedge Q] = \mathcal{P} \wedge \mathcal{L}[\mathcal{Q}] + (-1)^{pq}\mathcal{Q} \wedge \mathcal{L}[\mathcal{P}]. \tag{1.9}$$

**Lemma 1.3.** *IF* $\mathcal{L} \in \operatorname{Alt}^l_K A,\ \mathcal{P} \in \operatorname{Alt}^p_K A,\ \mathcal{Q} \in \operatorname{Alt}^q_K A$*, then*

$$(\mathcal{P} \wedge \mathcal{Q})[\mathcal{L}] = \mathcal{P}[\mathcal{L}] \wedge \mathcal{Q} + (-1)^{p(l-1)} \mathcal{P} \wedge \mathcal{Q}[\mathcal{L}]. \tag{1.10}$$

**Proposition 1.2.** *If* $\mathcal{L} \in \operatorname{Md}^l_K(A),\ \mathcal{P} \in \operatorname{Alt}^p_K A,\ \mathcal{Q} \in \operatorname{Alt}^q_K A$*, then*

$$\begin{aligned}[\![\mathcal{P} \wedge \mathcal{Q}, \mathcal{L}]\!]^{\text{RN}} &= \mathcal{P} \wedge [\![\mathcal{Q}, \mathcal{L}]\!]^{\text{RN}} + (-1)^{q(l-1)} [\![\mathcal{P}, \mathcal{L}]\!]^{\text{RN}} \wedge \mathcal{Q} \\ &= \mathcal{P} \wedge [\![\mathcal{Q}, \mathcal{L}]\!]^{\text{RN}} + (-1)^{qp} \mathcal{Q} \wedge [\![\mathcal{P}, \mathcal{L}]\!]^{\text{RN}}\end{aligned} \tag{1.11}$$

*Proof.* Using the definition of the Richardson−Nijenhuis bracket and the previous lemmas, we get

$$\begin{aligned}&[\![\mathcal{P} \wedge \mathcal{Q}, \mathcal{L}]\!]^{\text{RN}} = (-1)^{(p+q-1)(l-1)} (\mathcal{P} \wedge \mathcal{Q})[\mathcal{L}] - (\mathcal{L})[\mathcal{P} \wedge \mathcal{Q}] = \\ &(-1)^{(p+q-1)(l-1)} (\mathcal{P}[\mathcal{L}] \wedge \mathcal{Q} + (-1)^{pq} \mathcal{Q}[\mathcal{L}] \wedge \mathcal{P}) - \mathcal{P} \wedge \mathcal{L}[\mathcal{Q}] - (-1)^{pq} \mathcal{Q} \wedge \mathcal{L}[\mathcal{P}].\end{aligned} \tag{1.12}$$

Hence

$$-(-1)^{pq} \mathcal{Q} \wedge \mathcal{L}[\mathcal{P}] = -(-1)^{pq+q(l+p-1)} \mathcal{L}[\mathcal{P}] \wedge \mathcal{Q} = -(-1)^{q(l-1)} \mathcal{L}[\mathcal{P}] \wedge \mathcal{Q}.$$

Summing the first and fourth terms of (1.12), we have

$$\begin{aligned}&(-1)^{(p+q-1)(l-1)} (\mathcal{P}[\mathcal{L}] \wedge \mathcal{Q} - (-1)^{pq} \mathcal{Q} \wedge \mathcal{L}[\mathcal{P}] = \\ &\quad (-1)^{(p+q-1)(l-1)} (\mathcal{P}[\mathcal{L}] \wedge \mathcal{Q} - (-1)^{q(l-1)} \mathcal{L}[\mathcal{P}] \wedge \mathcal{Q} = \\ &\quad (-1)^{q(l-1)} \{(-1)^{(p-1)(l-1)} (\mathcal{P}[\mathcal{L}] \wedge \mathcal{Q} - \mathcal{L}[\mathcal{P}] \wedge \mathcal{Q}\} = \\ &\qquad (-1)^{q(l-1)} [\![\mathcal{P}, \mathcal{L}]\!]^{\text{RN}} \wedge \mathcal{Q} = (-1)^{qp} \mathcal{Q} \wedge [\![\mathcal{P}, \mathcal{L}]\!]^{\text{RN}}\end{aligned}$$

By a similar argument, summing the second and third terms of (1.12), we have

$$\begin{aligned}&(-1)^{(p+q-1)(l-1)+pq} \mathcal{Q}[\mathcal{L}] \wedge \mathcal{P}) - \mathcal{P} \wedge \mathcal{L}[\mathcal{Q}] = \\ &\qquad (-1)^{(q-1)(l-1)} \mathcal{P} \wedge \mathcal{Q}[\mathcal{L}] - \mathcal{P} \wedge \mathcal{L}[\mathcal{Q}] = \mathcal{P} \wedge [\![\mathcal{Q}, \mathcal{L}]\!]^{\text{RN}}.\end{aligned}$$

□

**Corollary 1.2.** *Under the conditions of the previous proposition, we have*

$$[\![\mathcal{L}, \mathcal{P} \wedge \mathcal{Q}]\!]^{\text{RN}} = [\![\mathcal{L}, \mathcal{P}]\!]^{\text{RN}} \wedge \mathcal{Q} + (-1)^{qp} [\![\mathcal{L}, \mathcal{Q}]\!]^{\text{RN}} \wedge \mathcal{P}. \tag{1.13}$$

**Proposition 1.3.** *If* $\mathcal{L}, \mathcal{N}, \mathcal{P}, \mathcal{Q} \in \operatorname{Md}^*_K(A)$*, then*

$$\begin{aligned}[\![\mathcal{P} \wedge \mathcal{Q}, \mathcal{L} \wedge \mathcal{N}]\!]^{\text{RN}} &= \mathcal{P} \wedge [\![\mathcal{Q}, \mathcal{L}]\!]^{\text{RN}} \wedge \mathcal{N} + (-1)^{ln} \mathcal{P} \wedge [\![\mathcal{Q}, \mathcal{N}]\!]^{\text{RN}} \wedge \mathcal{L} + \\ &(-1)^{ln+pq} \mathcal{Q} \wedge [\![\mathcal{P}, \mathcal{N}]\!]^{\text{RN}} \wedge \mathcal{L} + (-1)^{qp} \mathcal{Q} \wedge [\![\mathcal{P}, \mathcal{L}]\!]^{\text{RN}} \wedge \mathcal{N}.\end{aligned} \tag{1.14}$$

*Proof.* Using Proposition 1.2 and Corollary 1.2, we get

$$\begin{aligned}[\![\mathcal{P} \wedge \mathcal{Q}, \mathcal{L} \wedge \mathcal{N}]\!]^{\text{RN}} = &\mathcal{P} \wedge [\![\mathcal{Q}, \mathcal{L} \wedge \mathcal{N}]\!]^{\text{RN}} + (-1)^{pq} \mathcal{Q} \wedge [\![\mathcal{P}, \mathcal{L} \wedge \mathcal{N}]\!]^{\text{RN}} = \\ &\mathcal{P} \wedge [\![\mathcal{Q}, \mathcal{L}]\!]^{\text{RN}} \wedge \mathcal{N} + (-1)^{ln} \mathcal{P} \wedge [\![\mathcal{Q}, \mathcal{N}]\!]^{\text{RN}} \wedge \mathcal{L} + \\ (-1)^{ln+pq} &\mathcal{Q} \wedge [\![\mathcal{P}, \mathcal{N}]\!]^{\text{RN}} \wedge \mathcal{L} + (-1)^{qp} \mathcal{Q} \wedge [\![\mathcal{P}, \mathcal{L}]\!]^{\text{RN}} \wedge \mathcal{N}.\end{aligned}$$

□

Similar considerations applied to the case $\varphi \in \operatorname{Alt}^{|\varphi|}_K(A, K)$, $\psi \in \operatorname{Alt}^{|\psi|}_K(A, K)$, $\mathcal{L}, \mathcal{N} \in \operatorname{Alt}^*_K A$ lead to the following equalities:

$$\varphi[\mathcal{L}]_a = \varphi[\mathcal{L}_a] + (-1)^{l-1} \varphi_a[\mathcal{L}] \tag{1.15}$$

$$\mathcal{L}[\varphi \wedge \mathcal{N}] = \varphi \wedge \mathcal{L}[\mathcal{N}] \tag{1.16}$$

$$\varphi[\psi \wedge \mathcal{N}] = \psi \wedge \varphi[\mathcal{N}] \tag{1.17}$$

$$(\varphi \wedge \mathcal{N})[\mathcal{L}] = \varphi[\mathcal{L}] \wedge \mathcal{N} + (-1)^{|\varphi|(l-1)} \varphi \wedge \mathcal{N}[\mathcal{L}] \tag{1.18}$$

$$(\varphi \wedge \mathcal{N})_a = \varphi_a \wedge \mathcal{N} + (-1)^{|\varphi|} \varphi \wedge \mathcal{N}_a \tag{1.19}$$

Note that all these relations, except for (1.16) and (1.17), are similar to the corresponding ones for elements of $\mathrm{Alt}_K^* A$. Using these relations, it is easy to obtain the following formula:

$$\begin{aligned} [\![\varphi \wedge \mathcal{L}, \psi \wedge \mathcal{N}]\!]^{\mathrm{RN}} &= (-1)^{(|\varphi|+l-1)(|\psi|+n-1)} \psi \wedge \varphi[\mathcal{N}] \wedge \mathcal{L} - \varphi \wedge \psi[\mathcal{L}] \wedge \mathcal{N} \\ &+ \ (-1)^{|\psi|(l-1)} \varphi \wedge \psi \wedge [\![\mathcal{L}, \mathcal{N}]\!]^{\mathrm{RN}} \end{aligned} \tag{1.20}$$

## 2. Correlation between Richardson – Nijenhuis and Schouten brackets. The differential prolongation theorem

**Theorem 2.1.** *Suppose $M$ is a smooth manifold and $A = C^\infty(M)$; then the Richardson – Nijenhuis bracket $[\![\cdot,\cdot]\!]^{\mathrm{RN}}$ defined on the $A$-algebra of all skew symmetric multi-derivations $D_*(M) \subset \mathrm{Alt}_K^* A$ coincides with the Schouten bracket $[\![\cdot,\cdot]\!]^{\mathrm{SH}}$.*

*Proof.* Indeed, let $X$, $Y$ be a vector fields: $X,\ Y \in D(M)$. Then

$$[\![X, Y]\!]^{\mathrm{RN}} = [X, Y] = [\![X, Y]\!]^{\mathrm{SH}}.$$

Using formulas (1.11) and (1.13), one can express the value of Richardson – Nijenhuis bracket on any pair of decomposable vectors

$$X = X_1 \wedge \ldots \wedge X_n \in D_n(M),\ \ Y = Y_1 \wedge \ldots \wedge Y_l \in D_l(M)$$

via $[X_i, Y_j]$. For the Schouten bracket a similar role is played by formulas (a) and (b), [5, page 79]. But for the case in consideration (a) coincides with (1.11), and (b) coincides with (1.13). □

Denote by $\mathcal{F}(A^*)$, $A^* = \mathrm{Hom}_K(A, K)$, an arbitrary extension of the polynomial algebra over $A^*$ endowed with a flat connection in the sence of [14]. This means that any differential operator acting fron the polynomial algebra to $\mathcal{F}(A^*)$ has a unique prolongation to an operator $\mathcal{F}(A^*) \to \mathcal{F}(A^*)$. One can consider elements of $A$ as functions on $A^*$, that is the injection $A \subset \mathcal{F}(A^*)$ is defined. Using $K$-linearity and the Leibnitz rule, we can prolong any map $\mathcal{L} \in \mathrm{Alt}_K^l A$ to multi-differential operator $P_\mathcal{L}$ on $\mathcal{F}(A^*)$.

**Theorem 2.2.** *Let $\mathcal{L},\ \mathcal{N} \in \mathrm{Alt}_K^* A$. Then*

$$P_{[\![\mathcal{L},\mathcal{N}]\!]^{\mathrm{RN}}} = [\![P_\mathcal{L}, P_\mathcal{N}]\!]^{\mathrm{SH}}. \tag{2.1}$$

*Proof.* Let us consider $[\![\mathcal{L}, \mathcal{N}]\!]^{\mathrm{RN}}$. By definition,

$$P_{[\![\mathcal{L},\mathcal{N}]\!]^{\mathrm{RN}}}|_{A\subset\mathcal{F}(A^*)} = [\![\mathcal{L}, \mathcal{N}]\!]^{\mathrm{RN}}.$$

On the other hand, by the same definition and the definition of the Richardson – Nijenhuis bracket

$$[\![P_\mathcal{L}, P_\mathcal{N}]\!]^{\mathrm{RN}}|_{A\subset\mathcal{F}(A^*)} = [\![\mathcal{L}, \mathcal{N}]\!]^{\mathrm{RN}}.$$

At the same time, by Theorem 2.1,

$$[\![P_\mathcal{L}, P_\mathcal{N}]\!]^{\mathrm{RN}}|_{A\subset\mathcal{F}(A^*)} = [\![P_\mathcal{L}, P_\mathcal{N}]\!]^{\mathrm{SH}}|_{A\subset\mathcal{F}(A^*)}.$$

Finally, we note that the functionals $[\![P_\mathcal{L}, P_\mathcal{N}]\!]^{\mathrm{RN}}$ and $[\![P_\mathcal{L}, P_\mathcal{N}]\!]^{\mathrm{SH}}$ are uniquely determined by their values on $A \subset \mathcal{F}(A^*)$. □

## 3. Multi-algebra Lie structures

### 3.1. Main definition and first examples.

**Definition 3.1.** We shall say that a skew-symmetric map $\mathcal{L} \in \mathrm{Alt}^l_K A$ determines on $A$ a *Lie algebra structure of type* $(l,k,r)$, $0 \leqslant r \leqslant k < l$, (or simply an $(l,k,r)$-structure), if it satisfies the Jacobi identity of type $(l,k,r)$, i.e., for all $a_1,\dots,a_k,\ b_1,\dots b_r \in A$ we have

$$[\![\mathcal{L}_{b_1,\dots,b_r}, \mathcal{L}_{a_1,\dots,a_k}]\!]^{\mathrm{RN}} = 0 \tag{3.1}$$

For brevity, structures of type $(l,k,0)$ are called structures of type $(l,k)$. The set of all $(l,k,r)$-structures on $A$ is denoted by $\mathrm{L}^{(l,k,r)}(A)$, and the set of all $(l,k)$-structures is denoted by $\mathrm{L}^{(l,k)}(A)$.

**Example 3.1.** Let us consider $k = l-1$, $r = 0$. In this case (3.1) has the form

$$\mathcal{L}_{a_1,\dots,a_{l-1}}[\mathcal{L}] = \mathcal{L}[\mathcal{L}_{a_1,\dots,a_{l-1}}], \qquad \forall a_1,\dots,a_{l-1} \in A, \tag{3.2}$$

By definition, the left-hand side of this equality, applied to the set $b_1,\dots,b_l$, equals to

$$\mathcal{L}_{a_1,\dots,a_{l-1}}[\mathcal{L}](b_1,\dots,b_l) = \mathcal{L}_{a_1,\dots,a_{l-1}}(\mathcal{L}(b_1,\dots,b_l)) = \mathcal{L}(a_1,\dots,a_{l-1},\mathcal{L}(b_1,\dots,b_l)) = (-1)^{l-1}\mathcal{L}(\mathcal{L}(b_1,\dots,b_l),a_1,\dots,a_{l-1}).$$

Applying the right-hand side of (3.2) to the set $b_1,\dots,b_l$, we obtain

$$\mathcal{L}[\mathcal{L}_{a_1,\dots,a_{l-1}}](b_1,\dots,b_l) = \sum_{\substack{I+J=\mathrm{I}^{(l)}\\ |I|=1,\ |J|=l-1}} (-1)^{(I,J)}\mathcal{L}(\mathcal{L}_{a_1,\dots,a_{l-1}}(b_I),b_J) =$$

$$\sum_{(i+J)=\mathrm{I}^{(l)}} (-1)^{i-1}\mathcal{L}(\mathcal{L}_{a_1,\dots,a_{l-1}}(b_i),b_J) = \sum_{(i+J)=\mathrm{I}^{(l)}} (-1)^{i-1}\mathcal{L}(\mathcal{L}(a_1,\dots,a_{l-1},b_i),b_J) =$$

$$(-1)^{l-1}\sum_{(i+J)=\mathrm{I}^{(l)}} (-1)^{i-1}\mathcal{L}(\mathcal{L}(b_i,a_1,\dots,a_{l-1}),b_J) =$$

$$(-1)^{l-1}\sum_{i=1}^{l}\mathcal{L}(b_1,\dots,b_{i-1},\mathcal{L}(b_i,a_1,\dots,a_{l-1}),b_{i+1},\dots,b_l)$$

Also, (3.2) turns into the relation

$$\mathcal{L}(\mathcal{L}(b_1,\dots,b_l),a_1,\dots,a_{l-1}) = \sum_{i=1}^{l}\mathcal{L}(b_1,\dots,b_{i-1},\mathcal{L}(b_i,a_1,\dots,a_{l-1}),b_{i+1},\dots,b_l)$$

and we see that it is the Jacobi identity in the form (J1). So, $(l,l-1)$-structures are $l$-Lie algebra structures in the sense of Filippov [6].

**Example 3.2.** In our notations the Jacobi identity (J2) has the form

$$\mathcal{L}[\mathcal{L}] = 0. \tag{J2$'$}$$

Let $k = 0$, $r = 0$. In this case (3.1) has the form

$$[\![\mathcal{L},\mathcal{L}]\!]^{\mathrm{RN}} = (-1)^{l-1}\mathcal{L}[\mathcal{L}] - \mathcal{L}[\mathcal{L}] = 0,$$

If $l$ is even, then this condition coincides with (J2), i.e., in this case we obtain $l$-Lie algebra structures in the sense of [3, 9, 18]. Note, that the condition $[\![\mathcal{L},\mathcal{L}]\!]^{\mathrm{RN}} = 0$ is equivalent to the following one

$$[\![\mathcal{L},\mathcal{L}]\!]^{\mathrm{RN}}_a = -2[\![\mathcal{L},\mathcal{L}_a]\!]^{\mathrm{RN}} = 0 \quad \forall a \in A,\ \text{i. e. } [\![\mathcal{L},\mathcal{L}_a]\!]^{\mathrm{RN}} = 0 \quad \forall a \in A.$$

This means that $\mathrm{L}^{(2n,0)}(A) = \mathrm{L}^{(2n,1)}(A)$.

If $l$ is odd, then $[\![\mathcal{L},\mathcal{L}]\!]^{\mathrm{RN}} = \mathcal{L}[\mathcal{L}] - \mathcal{L}[\mathcal{L}] \equiv 0$, i.e. $\mathrm{L}^{(2n+1,0)}(A) = \mathrm{Alt}_K^{2n+1} A$. This case is trivial by itself, but it is interesting for a study of correlations between the structures of different types. It is easily seen that for odd $l$ (J2)-structures do not coincide with the structures in the sense of Definition 3.1.

3.2. **Heredity structures.** Recall (see, for example, [18]) that any $(l, l-1)$-structure $\mathcal{L}$ (in other words, (J1)-structure) generates a whole family of $(l-1, l-2)$-structures, because for all $a \in A$ the map $\mathcal{L}_a$ is an $(l-1, l-2)$-structure. In the general case a similar fact is valid too.

**Proposition 3.1.** *Let $\mathcal{L}$ be a Lie algebra of type $(l,k)$ and $k > 0$. Then for all $a \in A$ the map $\mathcal{L}_a$ is a Lie algebra structure of type $(l-1, k-1)$.*

*Proof.* Indeed, let $\mathcal{L} \in \mathrm{L}^{(l,k)}(A)$. Using (1.4), for all $a_1, \dots, a_k$ we obtain

$$[\![\mathcal{L}, \mathcal{L}_{a_1,\dots,a_k}]\!]^{\mathrm{RN}}_a = (-1)^{l-1}[\![\mathcal{L}_a, \mathcal{L}_{a_1,\dots,a_k}]\!]^{\mathrm{RN}} + [\![\mathcal{L}, \mathcal{L}_{a_1,\dots,a_k,a}]\!]^{\mathrm{RN}} = 0. \tag{3.3}$$

If $a = a_1$, then $[\![\mathcal{L}, \mathcal{L}_{a_1,\dots,a_k,a}]\!]^{\mathrm{RN}} = 0$ and therefore

$$[\![\mathcal{L}_a, \mathcal{L}_{a,a_2\dots,a_k}]\!]^{\mathrm{RN}} = 0.$$

This means that $\mathcal{L}_a$ is $(l-1, k-1)$-structure. □

**Definition 3.2.** Structures described by Proposition 3.1 are called *heredity structures*, associated with $\mathcal{L}$.

Relations between heredity structures of types $(l,k)$ are shown on Diagram 1.

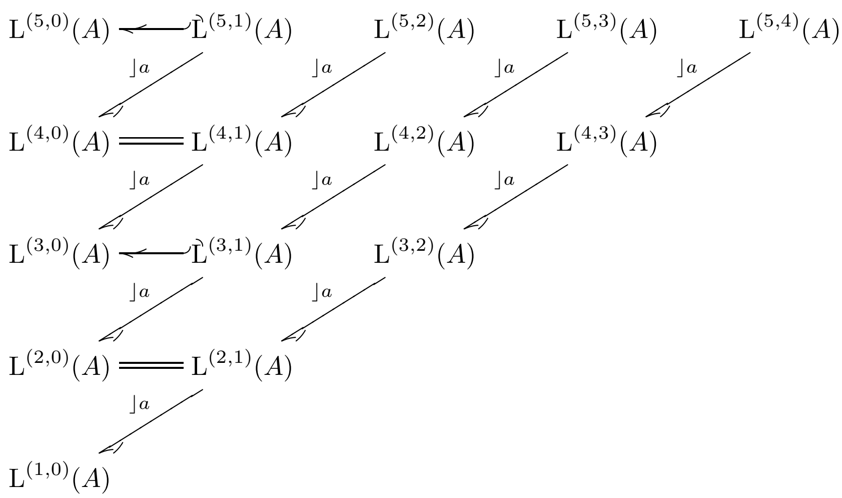


Diagram 1. Heredity structures

Suppose $A = C^\infty(M)$ and $\mathcal{L} \in \mathrm{L}^{(2,1)}(A)$ is bi-differential operator. In [13] A. A. Kirillov proved that the order of this operator is not greater than 1. In the paper [17], a similar result was proved for the case $\mathcal{L} \in \mathrm{L}^{(n,n-1)}(A)$. It follows from the Diagram 1 that it is valid for the case $\mathcal{L} \in \mathrm{L}^{(n,n-2)}(A)$ too. Actually, if $\mathcal{L} \in \mathrm{L}^{(n,n-2)}(A)$, then for all $a_1, \dots, a_{n-2} \in A$ the order of bi-differential operator $\mathcal{L}_{a_1,\dots,a_{n-2}} \in \mathrm{L}^{(2,1)}(A)$ is not greater than 1. This implies that the order of $\mathcal{L}$ is not greater than 1 too.

On the other hand, using the isomorphisms $\mathrm{L}^{(2n+1)}(A) = \mathrm{Alt}_K^{2n+1} A$ and taking into account Propositions 3.4, 3.5 and Theorems 3.5, 3.7, we see that generalization of the A. A. Kirillov theorem to arbitrary structures may not be valid.

**Proposition 3.2.** *Let $\mathcal{L} \in \mathrm{L}^{(l,k,r)}(A)$, where $r > 0$. Then*

$$\mathcal{L}_a \in \mathrm{L}^{(l-1,k,r-1)}(A) \bigcap \mathrm{L}^{(l-1,k-1,r)}(A) \bigcap \mathrm{L}^{(l-1,k-1,r-1)}(A), \text{ if } k > r,$$
$$\mathcal{L}_a \in \mathrm{L}^{(l-1,k,k-1)}(A) \bigcap \mathrm{L}^{(l-1,k-1,k-1)}(A), \text{ if } k = r.$$

*Proof.* The proof is a similar to that of Proposition 3.1. □

### 3.3. **Compatible structures.**

**Definition 3.3.** Two $(l, k, r)$-structures $\mathcal{L}, \mathcal{N}$ are said to be *compatible*, if $\alpha\mathcal{L}+\beta\mathcal{N}$ is an $(l, k, r)$-structure again for all $\alpha$, $\beta \in K$.

It follows immediately from the definition that structures $\mathcal{L}$ and $\mathcal{N}$ are compatible if and only if for all $a_1, \dots, a_k, b_1, \dots, b_r \in A$

$$[\![\mathcal{L}_{b_1,\dots,b_r}, \mathcal{N}_{a_1,\dots,a_k}]\!]^{\mathrm{RN}} + [\![\mathcal{N}_{b_1,\dots,b_r}, \mathcal{L}_{a_1,\dots,a_k}]\!]^{\mathrm{RN}} = 0. \tag{3.4}$$

This condition is a natural generalization of the Magri condition [16]. It is obvious that if $\mathcal{L} + \mathcal{N}$ is a Lie algebra structure, then $\mathcal{L}$ and $\mathcal{N}$ are compatible.

**Proposition 3.3.** *If $\mathcal{L}$ is a multi-Lie algebra structure, then the heredity structures $\mathcal{L}_a$ and $\mathcal{L}_b$ are compatible for all $a$, $b \in A$.*

*Proof.* Actually, in this case as in the (J1)-case (see, for example, [17]), we have $\mathcal{L}_a + \mathcal{L}_b = \mathcal{L}_{a+b}$. □

*Remark* 3.1. One can treat the Jacobi identity (3.1) as a self-compatibility condition.

### 3.4. **Exterior multiplication.**

The results of this subsection are proved by direct calculations, based on the formulas of Section 1. Additionally, all Lie algebra structures under consideration are assumed to be multi-derivations.

**Proposition 3.4.** *Let Lie algebra structures $\mathcal{L} \in \mathrm{L}^{(l,0)}(A)$, $\mathcal{N} \in \mathrm{L}^{(n,0)}(A)$. If $l+n$ is odd or $[\![\mathcal{L}, \mathcal{N}]\!]^{\mathrm{RN}} = 0$, then $\mathcal{L} \wedge \mathcal{N}$ is a Lie algebra structure of type $(l + n, 0)$.*

**Proposition 3.5.** *Let Lie algebra structures $\mathcal{L} \in \mathrm{L}^{(l,1)}(A)$, $\mathcal{N} \in \mathrm{L}^{(n,1)}(A)$ satisfy the condition $[\![\mathcal{L}, \mathcal{N}]\!]^{\mathrm{RN}} = 0$. If $l$ and $n$ have the same parity or $[\![\mathcal{L}, \mathcal{N}_a]\!]^{\mathrm{RN}} = 0$ for any $a \in A$, then $\mathcal{L} \wedge \mathcal{N}$ is a Lie algebra structure of type $(l + n, 1)$.*

**Theorem 3.4.** *Let $\mathcal{L}_i \in \mathrm{Alt}_K^1 A$, $i = 1, \dots, l$ and $\mathcal{L} = \mathcal{L}_1 \wedge \dots \wedge \mathcal{L}_l$.*

1. *Suppose $l > 3$, then $\mathcal{L}$ is multi Lie algebra structure of type $(l, 1)$.*
2. *Suppose $l = 3$, then $\mathcal{L}$ is $(3, 1)$-structure if and only if for all $a \in A$*

$$\mathcal{L}_1 \wedge \mathcal{L}_2 \wedge \mathcal{L}_3 \wedge ((\mathcal{L}_1)_a[\mathcal{L}_2, \mathcal{L}_3] - (\mathcal{L}_2)_a[\mathcal{L}_1, \mathcal{L}_2] + (\mathcal{L}_3)_a[\mathcal{L}_1, \mathcal{L}_2]) = 0.$$

*In particular, this condition holds, if the maps $\mathcal{L}_i$ commute with each other.*

### 3.5. **Coheredity structures.**

**Theorem 3.5.** *Let $\varphi \in \mathrm{Alt}_K^1(A, K)$, $\mathcal{L} \in \mathrm{L}^{(n,k)}(A)$ and $\varphi[\mathcal{L}] = 0$. Then $\varphi \wedge \mathcal{L} \in \mathrm{L}^{(n+1,k+1)}(A)$.*

*Proof.* It is easily seen that

$$(\varphi\wedge\mathcal{L})_{a_{\mathrm{I}(k+1)}}=\sum(-1)^{i-1}\varphi(a_i)\mathcal{L}_{a_{\mathrm{I}(k+1)\ -i}}+(-1)^{k+1}\varphi\wedge\mathcal{L}_{a_{\mathrm{I}(k+1)}} \tag{3.5}$$

Therefore

$$[\![\varphi\wedge\mathcal{L},(\varphi\wedge\mathcal{L})_{a_{\mathrm{I}(k+1)}}]\!]^{\mathrm{RN}}=$$
$$\sum(-1)^{i-1}\varphi(a_i)[\![\varphi\wedge\mathcal{L},\mathcal{L}_{a_{\mathrm{I}(k+1)\ -i}}]\!]^{\mathrm{RN}}+(-1)^{k+1}[\![\varphi\wedge\mathcal{L},\varphi\wedge\mathcal{L}_{a_{\mathrm{I}(k+1)}}]\!]^{\mathrm{RN}}$$

Using (1.20), we get

$$[\![\varphi\wedge\mathcal{L},\mathcal{L}_{a_{\mathrm{I}(k+1)\ -i}}]\!]^{\mathrm{RN}}=\pm\varphi[\mathcal{L}_{a_{\mathrm{I}(k+1)\ -i}}]\wedge\mathcal{L}+\varphi\wedge[\![\mathcal{L},\mathcal{L}_{a_{\mathrm{I}(k+1)\ -i}}]\!]^{\mathrm{RN}}.$$

Here the first term is equal to zero, because $\varphi[\mathcal{L}_{a_{\mathrm{I}(k+1)\ -i}}]=\varphi[\mathcal{L}]_{a_{\mathrm{I}(k+1)\ -i}}$, and the second one is zero, because $\mathcal{L}\in\mathrm{L}^{(n,k)}(A)$. Also,

$$[\![\varphi\wedge\mathcal{L},\varphi\wedge\mathcal{L}_{a_{\mathrm{I}(k+1)}}]\!]^{\mathrm{RN}}=$$
$$\pm\varphi\wedge\varphi[\mathcal{L}_{a_{\mathrm{I}(k+1)}}]\wedge\mathcal{L}-\varphi\wedge\varphi[\mathcal{L}]\wedge\mathcal{L}_{a_{\mathrm{I}(k+1)}}\pm\varphi\wedge\varphi\wedge[\![\mathcal{L},\mathcal{L}_{a_{\mathrm{I}(k+1)}}]\!]^{\mathrm{RN}}=0.$$

□

**Example 3.3.** Let $A=C^\infty(\mathbb{R}^n)$ and $x_1,\dots,x_n$ be a local coordinate system in $\mathbb{R}^n$,

$$\mathcal{L}=x_1\frac{\partial}{\partial x_1}\wedge\ldots\wedge\frac{\partial}{\partial x_k},\ \text{and}\ \varphi(f)=f(0,t_2,\dots,t_n),\ \ f\in A,\ \ t_i\in\mathbb{R}.$$

Then, by Theorem 3.4 $\mathcal{L}\in\mathrm{L}^{(k,1)}(A)$ and by the condition $\varphi[\mathcal{L}]=0$, one has $\varphi\wedge\mathcal{L}\in\mathrm{L}^{(k+1,1)}(A)$.

**Example 3.4.** Let $\mathcal{L}\in\mathrm{L}^{(1,0)}(A)=\mathrm{Alt}^1_K A$, $\varphi\in\mathrm{Alt}^1_K(A,K)$ and $\varphi\circ\mathcal{L}=0$. Then $\varphi\wedge\mathcal{L}$ is a standard Lie algebra structure.

**Definition 3.6.** Structures of type $\varphi\wedge\mathcal{L}$ are called *coheredity structures* associated with $\mathcal{L}$.

It follows from the equality

$$\varphi\wedge\mathcal{L}+\psi\wedge\mathcal{L}=(\varphi+\psi)\wedge\mathcal{L},\qquad\varphi,\ \psi\in\mathrm{Alt}^1_K(A,K),\ \ \mathcal{L}\in\mathrm{Alt}^l_K A,$$

that the structures $\varphi\wedge\mathcal{L}$ and $\psi\wedge\mathcal{L}$ are compatible. Suppose that the structures $\mathcal{L},\mathcal{N}\in\mathrm{Alt}^l_K A$ are compatible and $\varphi[\mathcal{L}]=\varphi[\mathcal{N}]=0$. Then the equality

$$\varphi\wedge\mathcal{L}+\varphi\wedge\mathcal{N}=\varphi\wedge(\mathcal{L}+\mathcal{N}),\qquad\varphi\in\mathrm{Alt}^1_K(A,K),$$

shows us that the structures $\varphi\wedge\mathcal{L}$ and $\varphi\wedge\mathcal{N}$ are compatible too.

Relations between coheredity structures of type $(l,k)$ implied by Theorem 3.5 are shown on Diagram 2.

**Theorem 3.7.** *Let*

1. $\mathcal{L}\in\mathrm{L}^{(l-1,k,r-1)}(A)\bigcap\mathrm{L}^{(l-1,k-1,r)}(A)\bigcap\mathrm{L}^{(l-1,k-1,r-1)}(A),\ \ k>r,$
   *or*
   $\mathcal{L}\in\mathrm{L}^{(l-1,k,r-1)}(A)\bigcap\mathrm{L}^{(l-1,k-1,r-1)}(A),\ \ k=r,$
2. $\varphi\in\mathrm{Alt}^1_K(A,K)$ *and* $\varphi[\mathcal{L}]=0$.

*Then* $\varphi\wedge\mathcal{L}\in\mathrm{L}^{(l+1,k,r)}(A)$.

Now let $\mathcal{L}\in\mathrm{L}^{(l,k,r)}(A)$, $k>0$, $\varphi\in\mathrm{Alt}^1_K(A,K)$ and $\varphi[\mathcal{L}]=0$. Taking into account Propositions 3.1, 3.2 and Theorems 3.5, 3.7, we obtain that for any $c\in A$ the map $\varphi\wedge\mathcal{L}_c$ belongs to $\mathcal{L}\in\mathrm{L}^{(l,k,r)}(A)$ again.

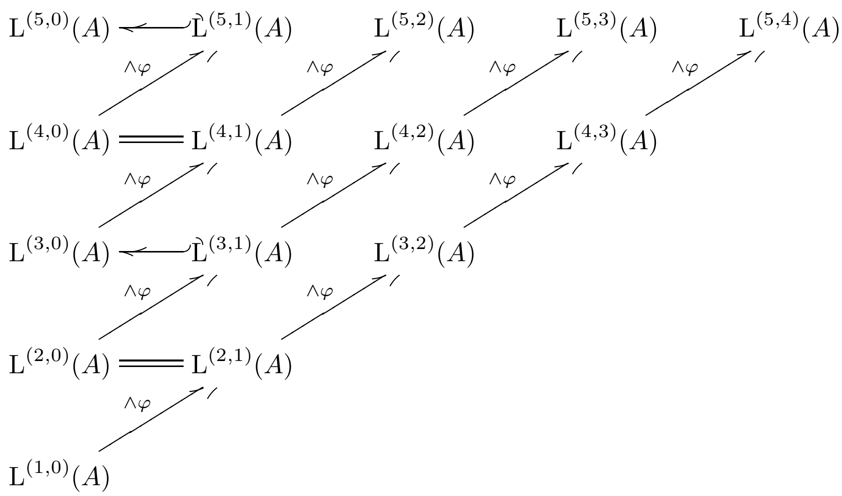


Diagram 2. Coheredity structures

**Theorem 3.8.** *Under the conditions of the previous paragraph the structures $\mathcal{L}$ and $\varphi \wedge \mathcal{L}_c$ are compatible.*

*Proof.* Combining (1.20), (3.5), and conditions of the theorem, we obtain

$$[\![\mathcal{L}_{b_{\mathrm{I}(r)}}, (\varphi \wedge \mathcal{L}_c)_{a_{\mathrm{I}(k)}}]\!]^{\mathrm{RN}} + [\![(\varphi \wedge \mathcal{L}_c)_{b_{\mathrm{I}(r)}}, \mathcal{L}_{a_{\mathrm{I}(k)}}]\!]^{\mathrm{RN}} = \\ \varphi \wedge \left((-1)^{l-r-1}[\![\mathcal{L}_{b_{\mathrm{I}(r)}}, \mathcal{L}_{a_{\mathrm{I}(k)},c}]\!]^{\mathrm{RN}} + [\![\mathcal{L}_{b_{\mathrm{I}(r)},c}, \mathcal{L}_{a_{\mathrm{I}(k)}}]\!]^{\mathrm{RN}}\right) = \\ \varphi \wedge [\![\mathcal{L}_{b_{\mathrm{I}(r)}}, \mathcal{L}_{a_{\mathrm{I}(k)}}]\!]_c^{\mathrm{RN}} = 0.$$

□

3.6. **Examples of ordinary structures.** Let $\varphi, \psi \in \mathrm{Alt}^1_K(A, K)$. Consider a structure $\mathcal{L} \in \mathrm{Alt}^{(l,k)}_K A$ satisfying $\varphi[\mathcal{L}] = \psi[\mathcal{L}] = 0$. Then, by (1.17),

$$\varphi[\psi \wedge \mathcal{L}] = \psi \wedge \varphi[\mathcal{L}] = 0$$

and, therefore, $\varphi \wedge \psi \wedge \mathcal{L} \in \mathrm{L}^{(l+2,k+2)}(A)$. This remark allows us to obtain examples of ordinary structures.

**Example 3.5.** Actually for an arbitrary $k$ the existence of $(k, 0)$- and $(k, 1)$-structures is guaranteed, in particular, by Propositions 3.4, 3.5 and by Theorem 3.4. To obtain an example of an $(l, n)$-structure, it is sufficient to take an $(l - n + 1, 1)$-structure $\mathcal{L}$ and choose elements

$$\varphi_1, \ldots, \varphi_{n-1} \in \mathrm{Alt}^1_K(A, K)$$

such that $\varphi_i[\mathcal{L}] = 0$. Then $\mathcal{N} \overset{\text{def}}{=} \varphi_1 \wedge \ldots \wedge \varphi_{n-1} \wedge \mathcal{L} \in \mathrm{L}^{(l,n)}(A)$.

In particular, suppose $A = C^\infty(\mathbb{R}^m)$, where $m$ is sufficiently large. Let $x_1, \ldots, x_m$ be a local coordinate system and $n < l \leqslant m$, then

$$\mathcal{L} = x_1 \ldots x_{n-1} \frac{\partial}{\partial x_n} \wedge \ldots \wedge \frac{\partial}{\partial x_l}$$

is a structure of type $(l - n + 1, 1)$. Define the maps $\varphi_i$, $1 \leqslant i \leqslant n - 1$, by the rules

$$\varphi_1(x_j) = \begin{cases} 1, & j = n-1, \\ 0, & j \neq n-1, \end{cases} \qquad \varphi_i(x_j) = \begin{cases} 1, & j = i-1, \\ 0, & j \neq i-1, \end{cases} \qquad i > 1.$$

Then $\varphi_i(\mathcal{L}) = 0$ and $\mathcal{N} = \varphi_1 \wedge \ldots \wedge \varphi_{n-1} \wedge \mathcal{L} \in \mathrm{L}^{(l,n)}(A)$. The structure $\mathcal{N}$ is not equal to zero, because $\mathcal{N}(x_n \wedge \ldots \wedge x_l) = 1$.

Set

$$\widehat{\mathrm{L}}^{(n,l)}(A) \stackrel{\mathrm{def}}{=} \bigcap_{i=1}^{l} \mathrm{L}^{(n,i)}(A).$$

Taking into account the embeddings $\mathrm{L}^{(2n+1,1)}(A) \subset \mathrm{L}^{(2n+1,0)}(A)$ and the isomorphisms $\mathrm{L}^{(2n,1)}(A) = \mathrm{L}^{(2n,0)}(A)$, we obtain that Lie algebra structure $\mathcal{N}$ constructed above in fact belongs to $\widehat{\mathrm{L}}^{(n,l)}(A)$ (see Diagram 2). Using this observation one can obtain examples of $(l,k,r)$-structures, where $r > 0$.

**Lemma 3.1.** *Let* $\mathcal{L} \in \mathrm{L}^{(l,k,r)}(A) \bigcap \mathrm{L}^{(l,k+1,r)}(A)$. *Then* $\mathcal{L} \in \mathrm{L}^{(l,k,r+1)}(A)$.

*Proof.* Taking into account the fact that $\mathcal{L} \in \mathrm{L}^{(l,k,r)}(A)$, we see that for any $b_{r+1} \in A$

$$[\![\mathcal{L}_{b_{\mathrm{I}(r)}}, \mathcal{L}_{a_{\mathrm{I}(k)}}]\!]^{\mathrm{RN}}_{b_{r+1}} = [\![(\mathcal{L}_{b_{\mathrm{I}(r)}})_{b_{r+1}}, \mathcal{L}_{a_{\mathrm{I}(k)}}]\!]^{\mathrm{RN}} + (-1)^{l-r-1)}[\![\mathcal{L}_{b_{\mathrm{I}(r)}}, (\mathcal{L}_{a_{\mathrm{I}(k)}})_{b_{r+1}}]\!]^{\mathrm{RN}} = 0$$

Note that $[\![\mathcal{L}_{b_{\mathrm{I}(r)}}, (\mathcal{L}_{a_{\mathrm{I}(k)}})_{b_{r+1}}]\!]^{\mathrm{RN}} = 0$ because $\mathcal{L} \in \mathrm{L}^{(l,k+1,r)}(A)$. Hence we obtain $[\![(\mathcal{L}_{b_{\mathrm{I}(r)}})_{b_{r+1}}, \mathcal{L}_{a_{\mathrm{I}(k)}}]\!]^{\mathrm{RN}} = 0$. □

**Proposition 3.6.** *Let us represent an integer* $n$ *as* $2k + \varepsilon$, *where* $\varepsilon$ *equals* $0$ *or* $1$ *in accordance with parity of* $n$. *Then*

$$\widehat{\mathrm{L}}^{(l,2k+\varepsilon)}(A) \subset \widehat{\mathrm{L}}^{(l,2k+\varepsilon-1,1)}(A) \subset \ldots \subset \widehat{\mathrm{L}}^{(l,k+\varepsilon,k)}(A).$$

In particular, this Proposition means that the $(l,n)$-structure $\mathcal{N}$ constructed in Example 3.5 really is a structure of type $(l, 2k+\varepsilon-s, s)$ for any $s \leqslant k$.

3.7. **Poisson multi-algebras.** Let $B$ be a vector space over $K$ and $\mathrm{Sym}^n_K A$, $\mathrm{Sym}^n_K B$ be sets of symmetric $n$-linear maps acting from $A$ to $A$ and from $B$ to $B$ respectively. Consider the maps

$$\mathcal{L} \in \mathrm{Alt}^n_K A,\ \mathcal{N} \in \mathrm{Alt}^n_K B$$

and $F \in \mathrm{Sym}^n_K A$, $G \in \mathrm{Sym}^n_K B$. One can put them into correspondence with the maps $\mathcal{L} \otimes G \in \mathrm{Alt}^n_K(A \otimes B)$ and $F \otimes \mathcal{N} \in \mathrm{Alt}^n_K(A \otimes B)$ by using the formulas

$$\begin{aligned} (\mathcal{L} \otimes G)(a_1 \otimes b_1, \ldots, a_n \otimes b_n) &\stackrel{\mathrm{def}}{=} \mathcal{L}(a_1, \ldots, a_n) \otimes G(b_1, \ldots, b_n) \\ (F \otimes \mathcal{N})(a_1 \otimes b_1, \ldots, a_n \otimes b_n) &\stackrel{\mathrm{def}}{=} F(a_1, \ldots, a_n) \otimes \mathcal{N}(b_1, \ldots, b_n) \end{aligned} \tag{3.6}$$

Now let $A$, $B$ be algebras over $K$. Define the maps $\Pi_A \in \mathrm{Sym}^n_K A$, $\Pi_B \in \mathrm{Sym}^n_K B$ by the rules $\Pi_A(a_1, \ldots, a_n) \stackrel{\mathrm{def}}{=} a_1 \cdot \ldots \cdot a_n$ and $\Pi_B(b_1, \ldots, b_n) \stackrel{\mathrm{def}}{=} b_1 \cdot \ldots \cdot b_n$ respectively.

**Definition 3.9.** Let $A$ be a $K$-algebra. We say that the map $\mathcal{L} \in \mathrm{Alt}^n_K A$ is a *Poisson algebra structure of type* $(n,k,r)$ on $A$ or $(n,k,r)$-Poisson structure, if

1. $\mathcal{L}$ is a Lie algebra structure of type $(n,k,r)$;
2. $\mathcal{L}$ is a multi-derivation, that is

$$\mathcal{L}(aa', a_2, \ldots, a_n) = a\mathcal{L}(a', a_2, \ldots, a_n) + a'\mathcal{L}(a, a_2, \ldots, a_n).$$

**Proposition 3.7.** *Let* $\mathcal{L} \in \mathrm{Alt}^n_K A$ *and* $\mathcal{N} \in \mathrm{Alt}^n_K B$ *be Lie* (*Poisson*) *structures of type* $(l,r)$. *Then maps* $\mathcal{L} \otimes \Pi_B$ *and* $\Pi_A \otimes \mathcal{N}$ *be Lie* (*Poisson*) *structures of type* $(l,r)$.

*Proof.* It is easy to check that

$$[\![\mathcal{L} \otimes \Pi_B, (\mathcal{L} \otimes \Pi_B)_{(a\otimes b)_I}]\!]^{\mathrm{RN}} = [\![\mathcal{L}, \mathcal{L}_{a_I}]\!]^{\mathrm{RN}} \otimes b_1 \cdot \ldots \cdot b_n.$$

Hence $\mathcal{L} \otimes \Pi_B$ is Lie (Poisson) structures of type $(l, r)$. □

**Proposition 3.8.** *Let $\mathcal{L} \in \mathrm{Alt}^n_K A$ and $\mathcal{N} \in \mathrm{Alt}^n_K B$ be $(n,1)$-Poisson structures and let $n$ be even. Then the $(n,1)$-Poisson structures $\mathcal{L} \otimes \Pi_B$ and $\Pi_A \otimes \mathcal{N}$ are compatible. In particular, $\mathcal{L} \oplus \mathcal{N} \stackrel{\mathrm{def}}{=} \mathcal{L} \otimes \Pi_B + \Pi_A \otimes \mathcal{N}$ is $(n,1)$-Poisson structure.*[1]

*Proof.* The proof is by direct but tedious calculation. □

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[1] If $n$ is odd, then this proposition is not valid.

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