INTRODUCTION TO THE DIFFERENTIAL CALCULUS OVER COMMUTATIVE ALGEBRAS: PROBLEMS

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Problem 1. When $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ is a field?

Problem 2. Construct an example of a non-associative ring.

Problem 3. Construct an example of an infinite field other than $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.

Problem 4. Describe ideals of $\mathbb{Z}_m$.

Problem 5. For finite-dimensional vector spaces L_1 and L_2 over a field F prove that $\operatorname{Hom}_F(L_1, L_2) \cong L_1^* \otimes L_2$.

Problem 6. Compute the module $\mathbb{Z}_3 \otimes_{\mathbb{Z}} \mathbb{Z}_6$.

Problem 7. Let $\{e_1, \dots, e_m\}$ and $\{f_1, \dots, f_n\}$ be free generators of free modules M and N respectively. Prove that $\{\, e_i \otimes f_j \mid i = 1, \dots, m \; j = 1, \dots, n \,\}$ are free generators of the module $M \otimes N$.

Problem 8. For free modules L and M of finite rank over a ring A prove that the map

$$\Phi \colon L^* \otimes M \to \operatorname{Hom}_A(L, M), \quad \Phi(\xi \otimes m)(l) = \xi(l) \cdot m, \qquad \xi \in L^*, \quad m \in M, \quad l \in L$$

is an isomorphism of modules over A.

Problem 9. Construct a category $\mathscr{C}$ such that the objects of $\mathscr{C}$ are algebras of smooth functions on smooth manifolds and for any two objects $C^\infty(M_1)$ and $C^\infty(M_2)$, where M_1 and M_2 are smooth manifolds, the set of morphisms $\mathscr{M}or(C^\infty(M_1), C^\infty(M_2))$ is a non-empty and non-zero subset of $\operatorname{Hom}(C^\infty(M_2), C^\infty(M_1))$.

Problem 10. Construct a functor F from the category of finite sets to the category of vector spaces over $\mathbb{R}$ such that for any finite set S the dimension of $F(S)$ equals the number of elements of S.

Problem 11. Prove that in the category of finite-dimensional vector spaces the functors $V \mapsto (V^*)^*$ and $V \mapsto V$ are equivalent.

Problem 12. Prove that in the category of all vector spaces the functors from the previous problem are not equivalent.

Problem 13. Prove that the composition of two covariant functors or two contravariant functors is a covariant functor. What is the composition of covariant and contravariant functors?

Problem 14. Prove that if every module over a ring A is free then A is a field.

To pass the exam, please, solve 20 problems which seem most interesting to you.

Problem 15. For any short exact sequence

$$0 \to L \to M \to N \to 0$$

of modules over a ring A and any module P over A construct the sequence

$$0 \to \mathrm{Hom}_A(P, L) \to \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N)$$

and prove that it is exact.

Problem 16. Prove that if P is a projective module then the sequence (see the previous problem)

$$0 \to \mathrm{Hom}_A(P, L) \to \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N) \to 0$$

is exact.

Problem 17. Prove that the Möbius bundle is nontrivial.

Problem 18. Let $a \in M$ be a point of a smooth manifold M. Prove that the ideal μ_a is projective iff $\dim M = 1$.

Problem 19. Describe the bundle that corresponds to μ_a if

(1) $M = \mathbb{R}$;
(2) $M = S^1$.

Problem 20. Is the module of derivations $D(C^\infty(K))$, where K is the coordinate cross in $\mathbb{R}^2$, projective?

Problem 21. Describe $D(A)$ if $A = C(\mathbb{R})$ the algebra of continuous functions.

Problem 22. Describe $D(A)$ if $A = C^m(\mathbb{R})$.

Problem 23. Describe $D(A)$ if $A = \mathbb{R}[x]/(x^n)$.

Problem 24. Describe $D(A)$ if $A = \mathbb{Z}_m[x]/(x^n)$.

Problem 25. Describe $D(A)$ if $A = \mathbb{Z}_2$.

Problem 26. Describe $\mathrm{Diff}_k(A, A)$ if $A = C^\infty(K)$, where K is the cross.

Problem 27. Describe $\mathrm{Diff}_k(A, A)$ if $A = \mathbb{R}[x]/(x^n)$.

Problem 28. Describe $\mathrm{Diff}_k(A, A)$ if $A = \mathbb{Z}_m[x]/(x^n)$.

Problem 29. Prove the formula

$$[\dots[[\Delta \circ \nabla, a_1], a_2] \dots, a_l] = \sum [\dots[[\Delta, a_{i_1}], a_{i_2}] \dots, a_{i_k}] \circ [\dots[[\nabla, a_{j_1}], a_{j_2}] \dots, a_{j_{l-k}}],$$

where Δ, $\nabla \in \mathrm{Hom}_K(P, Q)$ and the sum is over all indices such that $i_1 < i_2 < \dots < i_k$, $j_1 < j_2 < \dots < j_{l-k}$, $(\cup_r i_r) \cup (\cup_s j_s) = \{1, 2, \dots, l\}$.

Problem 30. Prove the formula

$$\Delta(a_1 \dots a_l p) = \sum_{k>0} (-1)^{k+1} a_{i_1} \dots a_{i_k} \Delta(a_{j_1} \dots a_{j_{l-k}} p),$$

where $\Delta \in \mathrm{Diff}_s(P, Q)$, $s < l$, and the sum is as in the previous problem.

Problem 31. Construct a natural isomorphism

$$\mathrm{Diff}^+(P, \mathrm{Diff}^+(Q, R)) = \mathrm{Diff}^+(P, \mathrm{Diff}(Q, R)).$$

Problem 32. Prove that the map $S\colon D(\mathrm{Diff}^+_{k-1}(P)) \to \mathrm{Diff}^+_k(P)$, $S(\nabla)(a) = \nabla(a)(1)$, is a differential operator of order 1.

Problem 33. Check in coordinate form that the mapping $\mathscr{D}_k\colon \mathrm{Diff}^+_k(A) \to A$ is a differential operator of order k for $A = C^\infty(\mathbb{R}^n)$.

Problem 34. Prove that the gluing map has the form $(c_{k,l}(\Delta))(a) = \Delta(a)(1)$.

Problem 35. Describe the ghost complex for the operator div: $D(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3)$, $(f_1, f_2, f_3) \mapsto \partial f_1/\partial x_1 + \partial f_2/\partial x_2 + \partial f_3/\partial x_3$. (You may take the Maxwell equation or, better, abelian 2-form gauge theory instead of the divergence.)

Problem 36. Prove that $\gamma_{k+1}\colon D_{k+1}(P) \to D_k(\mathrm{Diff}^+_1(P))$ is a differential operator of order 1.

Problem 37. Prove that the kernel of the composition

$$D_k(\mathrm{Diff}^+_1(P)) \to D_{k-1}(\mathrm{Diff}^+_1(\mathrm{Diff}^+_1(P))) \to D_{k-1}(\mathrm{Diff}^+_2(P))$$

is isomorphic to $D_{k+1}(P)$.

Problem 38. Prove that the composition $S_{k,l}$

$$D_k(\mathrm{Diff}^+_l(P)) \to D_{k-1}(\mathrm{Diff}^+_1(\mathrm{Diff}^+_l(P))) \to D_{k-1}(\mathrm{Diff}^+_{l+1}(P))$$

is a differential operator of order 1.

Problem 39. Check that $S_{k-1,l+1} \circ S_{k,l} = 0$.

Problem 40. Prove that $J^1(M) = M \times T^*(M)$.

Problem 41. Show that $de^x - e^x\,dx \neq 0$ if the module of 1-forms was defined in the category of all $C^\infty(\mathbb{R})$-modules.

Problem 42. Let $\mathscr{J}^k_{\mathrm{alg}}(P)$ be the module of jets of a module P in the category of all $C^\infty(M)$-modules. Assume that P is a projective finitely generated module. Prove that the jet module in the category of geometric modules has the form $\mathscr{J}^k_{\mathrm{geom}}(P) = \mathscr{J}^k_{\mathrm{alg}}(P) \big/ \bigcap_{a\in M} \mu_a \mathscr{J}^k_{\mathrm{alg}}(P)$.

Problem 43. Prove the following formula for the co-gluing map: $c^{k,l}(aj_{k+l}(p)) = aj_k(j_l(p))$.

Problem 44. Check that the wedge product $\wedge\colon \Lambda^k \otimes_A \Lambda^1 \to \Lambda^{k+1}$ is equal to the composition $\Lambda^k \otimes_A \Lambda^1 \xrightarrow{\lambda} \mathscr{J}^1(\Lambda^k) \xrightarrow{\psi^d} \Lambda^{k+1}$, where $\lambda(\omega \otimes da) = (-1)^k(j_1(a\omega) - aj_1(\omega))$.