

R-manifolds and multivalued solutions of PDE's

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- Jet manifolds and PDE.
 - Introduction: jet spaces $J^k(n, m)$.
 - Jets of vector bundles $J^k(\pi)$.
 - Jets of submanifolds $J^k(E, n)$.
 - Jets of fibring. Jets of mappings $M \rightarrow N$. Jets of functions.
 - Definitions of differential equation and its solution.

- Geometric interpretation of solutions. R-manifolds.
 - The Cartan distribution on $J^k(E, n)$.
 - Coordinate description of the Cartan distribution.
 - Geometric structure of the Cartan planes.
 - Description of integral submanifolds of the Cartan distribution.
 - Ray submanifolds, prolongations of integral submanifolds.
 - The structure of locally maximal integral submanifolds of the Cartan distribution.
 - R-manifolds.
 - Definitions of differential equation and its solution.

- Lie transformations (high order contact transformations).
 - Lie transformations. Point transformations.
 - Prolongations of point transformations. Prolongations of Lie transformations.
 - Prolongations of morphisms of contact structures.
 - Lie-Bäcklund theorem.

- Lie fields, liftings (prolongations) of Lie fields.
- Infinitesimal Lie-Bäcklund theorem.
- Extrinsic and intrinsic geometries of PDE.
 - External and internal points of view on PDE.
 - Problem of the reconstruction of the embedding $\mathcal{E} \rightarrow J^k$ and the Cartan distribution on J^k .
 - Extrinsic and intrinsic symmetries of PDE.
 - Rigidity. Examples.

Problems

1. Let $\pi : E^{n+m} \rightarrow M^n$ be a smooth bundle. Prove that $J^k(\pi)$ is an open everywhere dense subset of $J^k(E, n)$.
2. Let $\mathcal{E} \subset J^2(\mathbb{R}^3, 2)$ be the minimal surface equation. Prove that $\pi_{2,1} : \mathcal{E} \rightarrow J^1(\mathbb{R}^3, 2)$ is a nontrivial 2-dimensional vector bundle.
3. Let $k \geq l$. Prove that $J^k(E, n)$ is the manifold of $(k-l)$ -jets of submanifolds of the form $L^{(l)}$ in $J^l(E, n)$.
4. Prove that the Cartan distribution on $J^k(E, n)$ is locally determined by the set of the Cartan forms

$$\omega_\sigma^j = dp_\sigma^j - \sum_{i=1}^n dp_{\sigma+1_i}^j dx_i, \quad |\sigma| < k, \quad j = 1, \dots, m.$$

5. Let $\pi : E \rightarrow M$ be a fiber bundle and let $\pi_{1,0} : J^1(\pi) \rightarrow E$. Show that sections of the bundle $\pi_{1,0}$ are connections in the bundle π , while the condition of zero curvature determines a first order equation in the bundle $\pi_{1,0}$.
6. Consider the system of equations:

$$\begin{cases} u_x = f(x, y, u) \\ u_y = g(x, y, u). \end{cases}$$

Prove that if this system is compatible, then the Cartan distribution restricted to the corresponding surface is completely integrable.

7. Prove the following statements:
- (a) $L^{(k)}$ is a locally maximal integral manifold of the Cartan distribution.
 - (b) Let $Q \subset J^k(E, n)$ be an n -dimensional integral manifold that is transversal to the fibers of the projection $\pi_{k,k-1}$. Then, locally, Q is of the form $L^{(k)}$.
8. (a) Let $\xi = X(x, y, z)\frac{\partial}{\partial x} + Y(x, y, z)\frac{\partial}{\partial y} + Z(x, y, z)\frac{\partial}{\partial z}$ be a nonzero vector at a point $\theta \in J^0(2, 1)$, and P the straight line determined by this vector. Deduce equations describing the ray manifold $l(P)$.
- (b) Let W be a curve $x = \alpha(t), y = \beta(t), z = \gamma(t)$. Describe $\mathcal{L}(W)$.
- (c) Let W be a surface of the form $z = f(x, y)$. Describe $\mathcal{L}(W)$.
9. Prove that $\dim \mathcal{L}(W) = r + m\binom{k+n-r-1}{n-r-1}$, where $r = \dim W$.
10. Let $F : J^0(\pi) \rightarrow J^0(\pi)$ be a point transformation. Prove that if the lifting $F^{(1)}$ is defined in a neighborhood of a point $\theta \in J^1(\pi)$, then the lifting $F^{(k)}$ is defined in a neighborhood of any point θ' such that $\pi_{k,1}(\theta') = \theta$.
11. Prove that ordinary differential equations are non rigid.

Exercises

1. Prove that the family of neighborhoods $\pi_k^{-1}(U)$ together with the coordinate functions (x, p_σ^j) determines a smooth manifold structure in $J^k(\pi)$. Prove that $\dim J^k(\pi) = n + m\binom{n}{n+k}$
2. Prove that $\pi_k : J^k(\pi) \rightarrow M$ is a smooth locally trivial vector bundle.
3. Prove that $\pi_{k+1,k} : J^{k+1}(\pi) \rightarrow J^k(\pi)$ is a smooth locally trivial bundle (not vector bundle).
4. Prove that
 - (a) $\pi_{l,s} \circ \pi_{k,l} = \pi_{k,s}, k \geq l \geq s;$
 - (b) $\pi_l \circ \pi_{k,l} = \pi_k, k \geq l;$

- (c) $\pi_{k,l} \circ j_k(s) = j_l(s)$, $k \geq l$.
5. Prove that
- (a) $J^k(E, n)$ is a smooth manifold of dimension $n + m \binom{n}{n+k}$;
 - (b) $\pi_{k,l} : J^k(E, n) \rightarrow J^l(E, n)$, $k \geq l$ is a smooth locally trivial bundle;
 - (c) $\pi_{l,s} \circ \pi_{k,l} = \pi_{k,s}$, $\pi_{k,l} \circ j_k(L) = j_l(L)$, $k \geq l \geq s$.
6. Let $F : J^k(E, n) \rightarrow J^k(E, n)$ be a Lie transformation. Prove that
- (a) $\pi_{k+l, k+s} \circ F^{(l)} = F^{(s)} \circ \pi_{k+s, k+l}$, $l \geq s$;
 - (b) $\text{id}^{(s)} = \text{id}$;
 - (c) $(F \circ G)^{(s)} = F^{(s)} \circ G^{(s)}$.
7. Consider the Legendre transformation F in the space $J^1(2, 1)$: $\bar{x} = -p$, $\bar{y} = -q$, $\bar{u} = u - xp - yq$, $\bar{p} = x$, $\bar{q} = y$.
- (a) Prove that F is a Lie transformation;
 - (b) Describe the lifting $F^{(1)}$;
 - (c) Prove that F can not be represented in the form $F = G^{(1)}$, where G is a point transformation.
8. Let $F : E \rightarrow E'$ be a morphism over a diffeomorphism $\bar{F} : M \rightarrow M$ of two bundles π and π' over M . Define the prolongation $F^{(k)} : J^k(\pi) \rightarrow J^k(\pi')$ in such a way that $F^{(k)}$ will be a morphism of π_k in π'_k over \bar{F} .
9. Let $F : J^k(E, n) \rightarrow J^l(E, n)$, $k > l$ be a smooth surjection such that $F_* \mathcal{C}_\theta^{(k)} = \mathcal{C}_{F(\theta)}^{(l)}$. Define the prolongation $F^{(k)} : J^k(E, n) \rightarrow J^k(E, n)$.
10. For the vector field $X = x \frac{\partial}{\partial u} - u \frac{\partial}{\partial x}$ on $J^0(2, 1)$ find $X^{(2)}$.